Relative Bounded Cohomology for Groupoids

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Chapter 1

Introduction

Relative Bounded Cohomology for Groupoids

Bounded cohomology was originally introduced by Trauber and developed into a complex theory with numerous applications by Gromov in his groundbreaking article "Volume and bounded cohomology" [45]. We will illustrate now why this is an important invariant, linking topology, (Riemannian) geometry and (geometric) group theory.

The definition of bounded cohomology is rather simple: Consider a topological space X. Equip $C_*^{\mathrm{sing}}(X;\mathbb{R})$ with the ℓ^1-norm with respect to the basis $S_*(X)$ of singular simplices. Then, instead of taking the algebraic dual of $C_*^{\mathrm{sing}}(X;\mathbb{R})$ as in the definition of singular cohomology, consider the topological dual $B(C_*^{\mathrm{sing}}(X;\mathbb{R}),\mathbb{R})$. The cohomology of this cochain complex is denoted by $H_b^*(X;\mathbb{R})$ and is called the bounded cohomology of X. The operator norm on $B(C_*^{\mathrm{sing}}(X;\mathbb{R}),\mathbb{R})$ induces a semi-norm on $H_b^*(X;\mathbb{R})$ and this semi-norm is part of the definition of bounded cohomology.

This definition might seem to be just a minor variant of singular cohomology, but bounded cohomology and singular cohomology are in fact very different theories. For instance, bounded cohomology of S^1 vanishes while the bounded cohomology of $S^1 \vee S^1$ is non-trivial; in particular, bounded cohomology does not satisfy excision.

The importance of bounded cohomology for geometrical questions arises from its relation with the simplicial volume, originally introduced by Gromov in his proof of Mostow's rigidity theorem [45, 64]:

Definition. Let M be a compact, connected, oriented n-manifold, possibly with boundary. Let $[M,\partial M]_{\mathbb{R}} \in H_n(M,\partial M;\mathbb{R})$ be the real fundamental class. We call

$$\|M,\partial M\| := \inf\{\|a\|_1 \mid a \in C_n^{\mathrm{sing}}(M,\partial M;\mathbb{R}) \text{ is a real fundamental cycle of } M\}$$

the simplicial volume of $(M,\partial M)$. Here, $\|\cdot\|_1$ is the norm on $C_n^{\mathrm{sing}}(M,\partial M;\mathbb{R})$ induced from the ℓ^1-norm on $C_n^{\mathrm{sing}}(M;\mathbb{R})$ with respect to $S_n(M)$.

The simplicial volume is a homotopy invariant, but it encodes information about the Riemannian geometry of the manifold. The simplicial volume provides for example a lower bound for the minimal volume [45]. Furthermore, the simplicial volume is proportional to the Riemannian volume if M is a closed hyperbolic manifold [45, 74, 7]. Via the duality principle, the semi-norm on bounded cohomology can be used to calculate the simplicial volume [45, 20, 17]. We will explain this in more detail in Chapter 2.

Similar to group cohomology, bounded cohomology can also be defined combinatorially for groups, by taking the topological dual of the Bar resolution equipped with an appropriate norm. It turns out that this invariant is deeply related to geometric properties of groups. In particular, bounded cohomology of an amenable group is trivial [45], and this can be used to characterise amenable groups in terms of bounded cohomology [65, 49].

Even more astonishingly, bounded cohomology of a space basically only depends on the fundamental group of the space:

Theorem (The Mapping Theorem, [48, Theorem 4.1][45, 22])**.** *Let (X, x) be a pointed connected CW complex. Then the classifying map induces a canonical isometric isomorphism*

$$H_b^*(X; \mathbb{R}) \longrightarrow H_b^*(\pi_1(X, x); \mathbb{R})$$

of semi-normed graded $\mathbb{R}$-modules.

In particular, using the duality principle, one can directly deduce that the simplicial volume of a manifold of non-zero dimension with amenable fundamental group vanishes.

The mapping theorem and its applications are one cornerstone of the theory of bounded cohomology. In "Foundations of the theory of bounded cohomology" [48], Ivanov gives a very elegant proof of this theorem via relative homological algebra. First, Ivanov introduces strong, relatively injective resolutions for group modules. As group cohomology can be defined via the fundamental lemma of homological algebra using injective resolutions, bounded cohomology of a group can be calculated by strong, relatively injective resolutions via an appropriate fundamental lemma. Ivanov then demonstrates that $B(C_*^{\text{sing}}(\widetilde{X}), \mathbb{R})$ is a strong, relatively injective resolution of the trivial $\pi_1(X, x)$-module $\mathbb{R}$. Therefore, an isomorphism $H_b^*(X; \mathbb{R}) \longrightarrow H_b^*(\pi_1(X, x); \mathbb{R})$ is induced by the fundamental lemma. Care has to be taken with regard to the norms, but Ivanov shows that the induced maps are in this case indeed isometric isomorphisms.

In order to study for instance the simplicial volume of manifolds with boundary, one would like to have a relative version of the mapping theorem. Park [68] has extended the ideas of Ivanov to the relative case, but as noted by Frigerio and Pagliantini [40], Park's proof contains a serious gap. Until now, the closest to a relative version of the mapping theorem is the following result of Pagliantini:

Theorem ([66, Theorem 1])**.** *Let $i : A \hookrightarrow X$ be a CW-pair. Let X and A be connected and assume that $\pi_1(i)$ is injective and $\pi_k(i)$ an isomorphism for all $k \in \mathbb{N}_{>1}$. Then there exists a canonical isometric isomorphism*

$$H_b^*(X, A; \mathbb{R}) \longrightarrow H_b^*(\pi_1(X), \pi_1(A); \mathbb{R}).$$

We want to develop a version of the mapping theorem in the non-connected case in order to consider for example manifolds with non-connected boundaries. To do so, one has first to make sense of the "fundamental group" of a non-connected space. This can be achieved by considering fundamental groupoids instead of groups. Groupoids are natural generalisations of groups (and group actions). By definition, a groupoid is just a small category where each morphism is invertible, thus groupoids can be viewed as group-like structure where the composition is only partially defined. For us, important examples of groupoids will be families of groups and the fundamental groupoid, a straightforward generalisation of the fundamental group.

Our goals in this part of the thesis are:

- Define bounded cohomology combinatorially for (pairs of) groupoids in an accessible and straightforward fashion.
- Develop a version of relative homological algebra in the spirit of Ivanov for groupoids and pairs of groupoids. Derive a fundamental lemma in this setting, i.e., show that bounded cohomology of (pairs of) groupoids can be calculated by certain (pairs of) resolutions, generalising the concept of relatively injective resolutions for group modules.
- Use this to extend the mapping theorem to non-connected spaces.

Let $\mathcal{G}$ be a groupoid. We begin by defining bounded cohomology of groupoids via a straightforward generalisation $C_n(\mathcal{G})$ of the Bar resolution. We then develop relative homological algebra for groupoid modules and extend the definitions of strong and relatively injective resolutions into this context, derive a fundamental lemma for groupoid modules and we show:

Theorem (Bounded groupoid cohomology via relative homological algebra, Theorem 3.4.10)**.** *Let $\mathcal{G}$ be a groupoid and V a Banach $\mathcal{G}$-module. Furthermore, let $((D^*, \delta_D^*), \varepsilon\colon V \longrightarrow D^0)$ be a strong $\mathcal{G}$-resolution of V.*

Then for each strong cochain contraction of (D^*, ε) there exists a canonical norm non-increasing cochain map of this resolution to the standard resolution $(B(C_n(\mathcal{G}), V))_{n\in\mathbb{N}}$ of V extending id_V.

Ivanov showed that $B(C_*^{\mathrm{sing}}(\widetilde{X};\mathbb{R});\mathbb{R})$ is a strong relatively injective resolution of the trivial $\pi_1(X,x)$-module $\mathbb{R}$ if (X,x) is a pointed connected CW-complex. We construct a $\pi_1(X)$-version of this resolution for the fundamental groupoid $\pi_1(X)$ that also works for non-connected spaces. We show that this provides strong, relatively injective resolutions and deduce:

Corollary (Absolute Mapping Theorem for Groupoids, Corollary 5.2.21)**.** Let X be a CW-complex and let V be a Banach $\pi_1(X)$-module. Then there is a canonical isometric isomorphism of graded semi-normed $\mathbb{R}$-modules

$$H_b^*(X;V') \longrightarrow H_b^*(\pi_1(X);V')$$

Similarly, we define strong and relatively injective resolutions for pairs of groupoids in terms of appropriate pairs of resolutions, show a fundamental lemma and deduce that the bounded cohomology of pairs can be calculated by strong, relatively injective resolutions as well:

Corollary (Corollary 3.5.26)**.** Let $i\colon \mathcal{A} \longrightarrow \mathcal{G}$ be a groupoid pair and V a Banach $\mathcal{G}$-module. Let $(C^*, D^*, \varphi^*, (\nu, \nu'))$ be a strong, relatively injective $(\mathcal{G},\mathcal{A})$-resolution of V. Then there exists a canonical, semi-norm non-increasing isomorphism of graded $\mathbb{R}$-modules

$$H^*(C^*, D^*, \varphi^*) \longrightarrow H_b^*(\mathcal{G}, \mathcal{A}; V).$$

Using our version of relative homological algebra for groupoids, we then show the following relative version of the mapping theorem for groupoids, extending the result of Pagliantini to non-connected spaces:

Theorem (Relative Mapping Theorem, Theorem 5.3.11). *Let $i\colon A \hookrightarrow X$ be a CW-pair, such that i is π_1-injective (Remark 5.3.1) and induces isomorphisms between the higher homotopy groups on each connected component of A. Let V be a Banach $\pi_1(X)$-module. Then there is a canonical isometric isomorphism*

$$H_b^*(X, A; V') \longrightarrow H_b^*(\pi_1(X), \pi_1(A); V').$$

Finally, we give a definition of amenable groupoids that contains in particular the fundamental groupoid of spaces whose connected components have amenable fundamental groups. Similarly to the result in the group setting [65], we show that:

Corollary (Algebraic Mapping Theorem, Corollary 4.2.5). Let $i\colon \mathcal{A} \hookrightarrow \mathcal{G}$ be a pair of groupoids such that $\mathcal{A}$ is amenable. Let V be Banach $\mathcal{G}$-module. Then

$$H^n(j^*)\colon H_b^*(\mathcal{G}, \mathcal{A}; V') \longrightarrow H_b^*(\mathcal{G}; V')$$

is an isometric isomorphism for each $n \in \mathbb{N}_{\geq 2}$.

Outlook. Our definition of relative bounded cohomology is a straightforward generalisation of the group situation. We hope that it will be useful to study in particular relative (geometric) properties of groups via bounded cohomology in a more transparent fashion than before. For instance, one interesting task will be to give a more accessible proof of a characterisation of relatively hyperbolic groups in the spirit of Mineyev and Yaman [61] via relative groupoid cohomology.

Uniformly finite homology and cohomology

Uniformly finite homology is an exotic coarse homology theory, introduced by Block and Weinberger [10] to study large-scale properties of metric spaces. It is a quasi-isometry invariant and can thus be defined also for finitely generated groups, considering a word metric on the group.

One important property of uniformly finite homology is that the zero degree uniformly finite homology group $H_0^{\mathrm{uf}}(X; \mathbb{R})$ of a metric space X vanishes if and only if X is non-amenable [10]. Other applications include rigidity properties of metric spaces of bounded geometry [36, 78], the construction of aperiodic tilings for non-amenable spaces [10, 31] and results about the macroscopic dimension of manifolds [33, 34, 35]. For finitely generated groups, we show that uniformly finite homology is dual to bounded valued cohomology, which was introduced by Gersten to study hyperbolic groups with homological methods.

Uniformly finite homology groups are rather elusive. We are able to give a fairly concrete picture of classes in 0-degree uniformly finite homology, however. This is joint work with Francesca Diana [9]. Let G be a finitely generated infinite amenable group. Every invariant mean on G induces a linear function $H_0^{\mathrm{uf}}(G; \mathbb{R}) \longrightarrow \mathbb{R}$ and we write $\widehat{H}_0^{\mathrm{uf}}(G; \mathbb{R}) \subset H_0^{\mathrm{uf}}(G; \mathbb{R})$ for the intersection of the kernels of all maps induced by means and correspondingly call the classes in $\widehat{H}_0^{\mathrm{uf}}(G; \mathbb{R})$ *mean-invisible*. Since there are infinitely many distinct means, the quotient $H_0^{\mathrm{uf}}(G; \mathbb{R})/\widehat{H}_0^{\mathrm{uf}}(G; \mathbb{R})$ is infinite-dimensional, Proposition 6.7.2.

Let S be a Følner-sequence for G. We associate to each 0-cycle c a growth function $\beta_c^S\colon \mathbb{N} \longrightarrow \mathbb{R}$ that measures how fast the cycle growths with respect to the Følner-sequence. Cycles that grow faster than the boundary of S are non-trivial in $H_0^{\mathrm{uf}}(G;\mathbb{R})$ and cycles with distinct growth functions induce distinct classes in uniformly finite homology. We introduce a geometric criterion, *sparsity*, for 0-cycles to induce mean-invisible classes. By an explicit construction, we show that each growth function between the growth of the boundary and the growth of the full Følner-sequence can be realised as the growth function of a sparse cycle, thus:

Theorem (Theorem 6.7.19)**.** *Let G be a finitely generated infinite amenable group with a word metric. Then there is a Følner sequence S in G such that for each growth function $c\colon \mathbb{N} \longrightarrow \mathbb{R}_{>0}$ such that $c \prec 1$, there is a sparse subset $\Gamma \subset G$ such that $\beta_\Gamma^S \sim c$. In particular, there is an uncountable family of linear independent sparse classes in $H_0^{\mathrm{uf}}(G;\mathbb{R})$.*

Not much has been known about higher-degree uniformly finite homology. We will shortly sketch joint work with Francesca Diana about higher-degree uniformly finite homology. We have the following result about amenable groups:

Theorem ([9, Theorem 3.8], Theorem 6.6.3)**.** *Let G be a finitely generated amenable group. Let $H \leq G$ be an infinite index subgroup. For each $n \in \mathbb{N}$ such that the map*

$$H_n(i;\mathbb{R})\colon H_n(H;\mathbb{R}) \longrightarrow H_n(G;\mathbb{R})$$

induced by the inclusion $i\colon H \hookrightarrow G$ is non-trivial, $\dim_{\mathbb{R}} H_n^{\mathrm{uf}}(G;\mathbb{R}) = \infty$ *holds.*

We will discuss several applications of this result in Chapter 6.

Finally, after studying the relation between uniformly finite homology and quasi-morphisms, by using the result of Epstein and Fujiwara [37] about the bounded cohomology of 3-manifolds, we show the following:

Theorem (Theorem 6.6.9)**.** *Let M be a closed irreducible 3-manifold with fundamental group G. Then either G is finite or*

$$\dim_{\mathbb{R}} H_2^{\mathrm{uf}}(G;\mathbb{R}) = \infty.$$

Structure of the Thesis

The thesis is structured as follows.

In *Chapter 2*, we recall the definition of bounded cohomology and its most important properties. We mention some applications of bounded cohomology to geometric group theory and discuss the relation between bounded cohomology and simplicial volume.

In *Chapter 3*, we discuss basic properties of groupoids and introduce the fundamental groupoid. We then present the general setup for homological algebra in the groupoid setting. We define bounded cohomology and ℓ^1-homology with twisted coefficients for (pairs of) groupoids, generalising the definitions for groups. We define relatively injective and projective strong resolutions and show how they can be used to calculate bounded cohomology and ℓ^1-homology respectively. We also give a definition of pairs of resolutions that calculate relative bounded cohomology.

Then, in *Chapter 4*, we give a definition of amenable groupoids, show that this property is characterised by bounded cohomology and prove the algebraic mapping theorem for groupoids.

In *Chapter 5*, we associate to a CW-complex X a $\pi_1(X)$-cochain complex and use it to define bounded cohomology of X with twisted coefficients in a module V over the fundamental groupoid, generalising the usual definition for connected spaces. We show that this cochain complex is a strong, relatively injective resolution of V and derive the absolute mapping theorem. Finally, we show that for certain CW-pairs this construction leads to the relative mapping theorem.

Finally, in *Chapter 6*, we discuss uniformly finite homology. We give a quite concrete picture of the classes in zero degree uniformly finite homology, differentiating in particular classes that can be detected by means and classes invisible to means and we show that there are infinitely many of both types, giving an explicit construction in the later case. We also present several calculations of higher degree uniformly finite homology.

In the *Appendix*, we sketch the arguments of Ivanov and Pagliantini regarding the construction of the strong cochain contractions necessary for the proof of the absolute and relative mapping theorem respectively.

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Chapter 2

Bounded Cohomology

In this chapter, we will give a short overview of bounded cohomology. We begin by presenting the definition of bounded cohomology of spaces and groups. The generalisation of this concept to groupoids will be central in the next chapters. We will then recall some well-known properties and applications of bounded cohomology, some of which will also be later generalised to the groupoid setting.

2.1 Bounded Cohomology of Groups and Spaces

Definition 2.1.1.

(i) A normed $\mathbb{R}$-chain complex $(C_*, \|\cdot\|)_{*\in\mathbb{Z}}$ is a chain complex of normed $\mathbb{R}$-modules, such that the boundary maps are bounded linear functions.

(ii) Let G be a group. A normed G-chain complex is a normed $\mathbb{R}$-chain complex together with a G-action by chain maps, such that the action in each degree is isometric.

Similarly, we also define normed (G-)cochain complexes.

Remark 2.1.2. If $(C_*, \|\cdot\|)$ is a normed chain complex, for each $n \in \mathbb{N}$, we get an induced semi-norm on the homology $H_n(C_*)$ by setting for each $\alpha \in H_n(C_*)$

$$\|\alpha\| := \inf\{\|a\| \mid a \in C_n,\ \partial_n a = 0,\ [a] = \alpha\}.$$

Similarly we also define a semi-norm for cochain complexes.

Definition 2.1.3. Let X be a topological space.

(i) We endow the singular chain complex $C_*^{\text{sing}}(X;\mathbb{R})$ with the ℓ^1-norm with respect to the basis $S_*(X)$ of singular simplices. Then $C_*^{\text{sing}}(X;\mathbb{R})$ is a normed $\mathbb{R}$-chain complex.

(ii) We write $C_b^*(X;\mathbb{R}) := B(C_*^{\text{sing}}(X;\mathbb{R}),\mathbb{R})$ for the dual cochain complex endowed with the $\|\cdot\|_\infty$-norm. Here, B denotes the space of bounded linear functions.

(iii) We call $H_b^*(X;\mathbb{R}) := H^*(C_b^*(X;\mathbb{R}))$, endowed with the induced semi-norm, *the bounded cohomology of X with coefficients in* $\mathbb{R}$. This defines a functor $\mathsf{Top} \longrightarrow \mathbb{R}\text{-}\mathsf{Mod}_*^{\|\cdot\|}$.

Here, we write Top to denote the category of topological spaces and $\mathbb{R}\text{-}\mathsf{Mod}_*^{\|\cdot\|}$ for the category of semi-normed graded $\mathbb{R}$-modules together with graded bounded linear maps. Similarly, we define a relative version of bounded cohomology:

Definition 2.1.4. Let $i\colon A \hookrightarrow X$ be a pair of topological spaces. Then we write $C_b^*(X, A; \mathbb{R})$ for the kernel of the map

$$B(C_*^{\text{sing}}(i; \mathbb{R}), \mathbb{R})\colon B(C_*^{\text{sing}}(X; \mathbb{R}), \mathbb{R}) \longrightarrow B(C_*^{\text{sing}}(A; \mathbb{R}), \mathbb{R}),$$

together with the norm induced by the norm on $B(C_*^{\text{sing}}(X; \mathbb{R}), \mathbb{R})$. This is a normed cochain complex and we call

$$H_b^*(X, A; \mathbb{R}) := H^*(C_b^*(X, A; \mathbb{R})),$$

endowed with the induced semi-norm, *the bounded cohomology of X relative to A with coefficients in* $\mathbb{R}$.

It is not difficult to see that bounded cohomology is a homotopy invariant of topological spaces. Bounded cohomology might appear to be a straightforward functional analytical variant of singular cohomology, but its behaviour is indeed very different from the behaviour of singular cohomology. This can be seen already in the following examples:

Example 2.1.5.

(i) We have $H_b^n(S^1; \mathbb{R}) = 0$ for all $n \in \mathbb{N}_{>0}$. See the next example.

(ii) If X is a simply connected space, or more generally, if X is connected and $\pi_1(X, x)$ is amenable for some $x \in X$, then for all $n \in \mathbb{N}_{>0}$ we have $H_b^n(X; \mathbb{R}) = 0$, [45, 48]. We will prove this more generally for groupoids in Section 5.3.2.

(iii) On the other hand $H_b^2(S^1 \vee S^1; \mathbb{R})$ is infinite dimensional [13].

Remark 2.1.6. In particular, bounded cohomology does *not* satisfy excision.

For many applications it will be useful to consider more generally bounded cohomology with twisted coefficients:

Remark 2.1.7. Let X be a connected CW-complex, $x \in X$ and V a Banach $\pi_1(X, x)$-module, i.e., a Banach $\mathbb{R}$-module with an isometric $\pi_1(X, x)$-action. Then $C_*^{\text{sing}}(\widetilde{X}; \mathbb{R})$ is a normed $\pi_1(X, x)$-chain complex, and we set

$$C_b^*(X; V) := B_{\pi_1(X,x)}(C_*^{\text{sing}}(\widetilde{X}; \mathbb{R}), V).$$

Here, $B_{\pi_1(X,x)}$ denotes the space of $\pi_1(X, x)$-equivariant bounded linear functions. Together with the $\|\cdot\|_\infty$-norm, this is a normed chain complex.

Definition 2.1.8. Let X be a connected CW-complex, $x \in X$ and V a Banach $\pi_1(X, x)$-module. We call

$$H_b^*(X; V) := H^*(C_b^*(X; V)),$$

endowed with the induced semi-norm, *the bounded cohomology of X with coefficients in V*.

We will extend this definition to non-connected spaces and twisted coefficients in Chapter 5.

Definition 2.1.9. Let X be a connected CW-complex, $x \in X$ and V a Banach $\pi_1(X,x)$-module. The canonical inclusion $C_b^*(X;V) \longrightarrow C^*(X;V)$ induces a map

$$c_{b,V}^* \colon H_b^*(X;V) \longrightarrow H^*(X;V),$$

called *the comparison map (with respect to coefficients in V).*

As we can see already from Example 2.1.5, the comparison map is in general neither injective nor surjective and determining when one of this properties holds is an area of active research. Injectivity in degree 2 is for instance related to stable commutator length [5] and to quasi-morphisms [37, 13, 14], while surjectivity can be used to describe hyperbolic groups (Theorem 2.2.2).

As for singular cohomology, one can express the bounded cohomology of the classifying space BG of a group G combinatorially in terms of the group:

Remark 2.1.10. Let G be a group.

(i) For each $n \in \mathbb{N}$, we write $P_n(G) := G^{n+1}$ and set

$$L_n(G) := \mathbb{R}\langle P_n(G)\rangle := \bigoplus_{P_n(G)} \mathbb{R},$$

endowed with the ℓ^1-norm with respect to $P_n(G)$. For all $k \in \mathbb{Z}_{<0}$, we set $C_k(G) = 0$.

(ii) We define boundary maps by setting for each $n \in \mathbb{N}_{>0}$

$$\begin{aligned}\partial_n \colon C_n(G) &\longrightarrow C_{n-1}(G)\\ (g_0,\dots,g_n) &\longmapsto \sum_{i=0}^{n}(-1)^i \cdot (g_0,\dots,\hat{g}_i,\dots,g_n).\end{aligned}$$

and by setting $\partial_k = 0$ for all $k \in \mathbb{Z}_{\leq 0}$. Then L_* together with the boundary maps ∂_* is a normed G-chain complex.

Definition 2.1.11. Let G be a group and V a Banach G-module. We call

$$H_b^*(G;V) := H^*(B_G(L_*(G),V))$$

the bounded cohomology of G with coefficients in V

Similar to the result about singular cohomology, one has:

Proposition 2.1.12. *There is an isometric isomorphism*

$$H_b^*(BG;\mathbb{R}) \longrightarrow H_b^*(G;\mathbb{R})$$

of semi-normed graded $\mathbb{R}$-modules.

One astonishing property of bounded cohomology is, that, in sharp contrast to singular cohomology, it basically only depends on the fundamental group:

Theorem 2.1.13 (The Mapping Theorem, [45],[48, Theorem 4.1])**.** *Let (X, x) be a pointed connected countable CW complex. Then there is a canonical isometric isomorphism*

$$H_b^*(X; \mathbb{R}) \longrightarrow H_b^*(\pi_1(X, x); \mathbb{R})$$

of semi-normed graded $\mathbb{R}$-modules.

We will sketch Ivanov's proof of the mapping theorem in Appendix A. Using averaging techniques on the group side, this theorem implies in particular the vanishing of bounded cohomology of spaces having amenable fundamental groups.

The mapping theorem in the groupoid setting will be discussed in Chapter 5. In particular, we will prove a relative version of the mapping theorem for certain pairs of not necessarily connected spaces (Theorem 5.3.11).

2.2 Applications

2.2.1 Bounded Cohomology and Geometric Properties of Groups

Bounded cohomology of finitely generated groups is *not* a quasi-isometry invariant [24, Corollary 1.7]. It demonstrates however, a deep relation with geometric concepts in group theory. As we will discuss now, it detects for instance both amenability and hyperbolicity of groups.

The following theorem was proven by Noskov:

Theorem 2.2.1 ([65])**.** *Let G be a group. Then the following are equivalent:*

(i) The group G is amenable.

(ii) For all Banach G-modules V and all $n \in \mathbb{N}_{>0}$, we have $H_b^n(G; V') = 0$.

(iii) For all Banach G-modules V, we have $H_b^1(G; V') = 0$.

Here, V' denotes the topological dual of V.

We will discuss an extension of this result to groupoids in Chapter 4. The following result is due to Mineyev:

Theorem 2.2.2 ([58, 59])**.** *Let G be a finitely presented group. Then the following are equivalent:*

(i) The group G is hyperbolic.

(ii) The comparison map $c_{b,V}^2 \colon H_b^2(G; V) \longrightarrow H^2(G; V)$ is surjective for any normed G-module V.

(iii) The comparison maps $c_{b,V}^n \colon H_b^n(G; V) \longrightarrow H^n(G; V)$ are surjective for any $n \in \mathbb{N}_{\geq 2}$ and any normed G-module V.

We will discuss this result and a similar result for bounded valued cohomology briefly in Section 6.3.

2.2.2 Simplicial Volume

In this section, we will briefly discuss the simplicial volume of compact, connected, oriented manifolds. We will present some glimpses as to why this is a significant invariant, linking topology and (Riemannian) geometry. We mention how bounded cohomology can often be used to derive information about simplicial volume, thus also explaining one reason why the semi-norm on bounded cohomology is of principal importance.

Let (X, A) be a pair of topological spaces. As we have seen, $C_*^{\mathrm{sing}}(X;\mathbb{R})$, together with the ℓ^1-norm with respect to $S_*(X)$, is a normed chain complex and this induces a norm turning $C_*^{\mathrm{sing}}(X, A;\mathbb{R})$ into a normed chain complex. We call the induced semi-norm $\|\cdot\|_1$ on $H_*(X;\mathbb{R})$ and $H_*(X, A;\mathbb{R})$ respectively *the ℓ^1-norm on $H_*(X;\mathbb{R})$ and $H_*(X, A;\mathbb{R})$ respectively.*

Definition 2.2.3. Let M be a compact, connected, oriented n-manifold, possibly with boundary. Let $[M, \partial M]_\mathbb{R} \in H_n(M, \partial M;\mathbb{R})$ be the real fundamental class, i.e., the image of the fundamental class $[M, \partial M]_\mathbb{Z} \in H_n(M, \partial M;\mathbb{Z})$ under the change-of-coefficients map $H_n(M, \partial M;\mathbb{Z}) \longrightarrow H_n(M, \partial M;\mathbb{R})$. We call

$$\|M, \partial M\| := \|[M, \partial M]_\mathbb{R}\|_1$$

the simplicial volume of M.

By definition, the simplicial volume is a homotopy invariant for compact, connected, oriented manifolds.

Theorem 2.2.4 (Proportionality Principle)**.** *Let M be a closed, connected, oriented manifold. Then there is a constant $c(\widetilde{M}) \in \mathbb{R}_{\geq 0}$, depending only on the Riemannian universal cover of M, such that*

$$\|M\| = \mathrm{Vol}(M) \cdot c(\widetilde{M})$$

For hyperbolic manifolds, $c(\mathbb{H}^n) = 1/\nu_n$, where ν_n is the volume of any regular ideal simplex in $\overline{\mathbb{H}}^n$. In particular, $c(\mathbb{H}^n) > 0$. In general, however, the constant $c(\widetilde{M})$ might be zero.

The proportionality principle result goes back to Gromov [45], who proved it using bounded cohomology and Thurston [74, Theorem 6.2.2], who described a different proof via measure homology. Gromov's proof has been worked out in detail by Bucher-Karlsson and Frigerio [21, 39]. Thurston's proof has been completed by Löh [53], also using bounded cohomology via the duality principle, Proposition 2.2.7.

Remark 2.2.5. The proportionality principle for hyperbolic manifolds also plays an important role in Gromov's proof of the Mostow rigidity theorem [64, 7].

We will see now, how bounded cohomology relates to the ℓ^1-norm on singular homology. First, there is a duality pairing:

Definition 2.2.6 (Kronecker Product)**.** The evaluation map

$$C_*^{\mathrm{sing}}(X) \otimes C_b^*(X;\mathbb{R}) \longrightarrow \mathbb{R}$$

induces a well-defined $\mathbb{R}$-map

$$\langle\,\cdot\,,\,\cdot\,\rangle\colon H_*(X;\mathbb{R}) \otimes H_b^*(X;\mathbb{R}) \longrightarrow \mathbb{R}$$

called *the Kronecker product.*

By the Hahn-Banach theorem, one gets:

Proposition 2.2.7 (Duality Principle for the ℓ^1-Norm, [45, Section 1.1])**.** *Let X be a topological space. Then for all $\alpha \in H_n(X;\mathbb{R})$, we get*

$$\|\alpha\|_1 = \sup\left\{ \frac{1}{\|\varphi\|_\infty} \;\middle|\; \varphi \in H_b^n(X;\mathbb{R}),\ \langle \alpha, \varphi\rangle = 1 \right\} \cup \{0\} \in \mathbb{R}_{\geq 0}.$$

Thus, we can use bounded cohomology to study simplical volume [45, 20, 17].

Corollary 2.2.8 ([45, 58, 59])**.** If the fundamental group of a closed, connected, oriented, aspherical (or more generally: rationally essential) n-manifold M is hyperbolic and $n \in \mathbb{N}_{\geq 2}$, the simplicial volume $\|M\|$ is positive.

Proof. Rationally essential implies that the image of the fundamental class $[M]_\mathbb{R}$ of M under the map $c_n\colon H_n(M;\mathbb{R}) \longrightarrow H_n(B\pi_1(M,m);\mathbb{R})$ induced by the classifying map is not trivial. Thus, there is a class $\alpha \in H^n(\pi_1(M,m);\mathbb{R})$, such that $\langle c_n([M]_\mathbb{R}), \alpha\rangle = 1$. Since $\pi_1(M,m)$ is a hyperbolic group, by Theorem 2.2.2 the comparison map $H_b^n(\pi_1(M,m);\mathbb{R}) \longrightarrow H^n(\pi_1(M);\mathbb{R})$ is surjective, hence there is also a class $\beta \in H_b^n(\pi_1(M);\mathbb{R})$, such that $\langle c_n([M]_\mathbb{R}), \beta\rangle = 1$ and it follows from the duality principle that $\|M\| > 0$. ◻

Corollary 2.2.9. Let M be a closed, connected, oriented n-manifold, such that the fundamental group of M is amenable. Then

$$\|M\| = 0.$$

In general, explicit formulas for non-vanishing simplicial volume are very rare. As we have seen, the simplicial volume is known (in terms of the volume) for hyperbolic manifolds. It is also additive with respect to certain gluing constructions along amenable boundaries [45, 72, 18] and, if the dimension is at least 3, with respect to connected sums [45, Section 3.5]. The principal example not arising from applying these constructions to hyperbolic manifolds is the calculation of the simplicial volume of manifolds covered by $\mathbb{H}^2 \times \mathbb{H}^2$ by Bucher-Karlsson:

Theorem 2.2.10 ([21])**.** *Let M be a closed Riemannian manifold, whose Riemannian universal cover is isometric to $\mathbb{H}^2 \times \mathbb{H}^2$. Then*

$$\|M\| = \frac{3}{2\cdot\pi^2} \cdot \operatorname{Vol}(M).$$

Also in this example, bounded cohomology plays an important part in the proof.

2.2.3 Other Applications

We end this chapter by listing some further applications of bounded cohomology, with no attempt at completeness:

- Various types of superrigidity results [63, 60, 27, 8].
- Generalised Milnor-Wood-type inequalities [23].
- Volume-rigidity for representations of hyperbolic lattices which implies in particular the Mostow-Prasad rigidity theorem in the case of hyperbolic lattices [19].
- Bounded cohomology in degree 2 detects non-trivial quasi-morphisms [37, 13, 14].
- Quasi-isometry classification of certain central extensions of $\mathbb{Z}$ [42].

Chapter 3

Bounded Cohomology for Groupoids

In this chapter, we introduce bounded cohomology for (pairs of) groupoids. In the first section, we recall the definition of groupoids, give some basic examples and repeat general facts about groupoids. In particular, we discuss the fundamental groupoid of a topological space, generalising the fundamental group.

In the second section, we present the general setup of homological algebra necessary to deal with the groupoid setting, similar to the group case. Specifically, we consider the Bar resolution and define groupoid (co-)homology, preparing the ground for our definition of bounded cohomology of groupoids.

In the third section, we introduce bounded cohomology and ℓ^1-homology with coefficients for groupoids, generalising the definition for groups, and derive some fundamental properties of these concepts.

Next, in the fourth section, we develop the setting to deal with bounded cohomology and ℓ^1-homology of groupoids via appropriate resolutions. In particular, we discuss the fundamental lemma of homological algebra in this setting.

In the last section, we define bounded cohomology for pairs of groupoids via the pair of standard resolutions and discuss how other pairs of resolutions can be used to study bounded cohomology of pairs.

3.1 Groupoids

Groupoids are a generalisation of groups (and group actions), akin to considering not necessarily connected spaces in topology. They can be viewed as group-like structures where composition is only partially defined.

Among many other applications, groupoids arise naturally in topology, e.g. in the form of the fundamental groupoid. This generalisation leads directly to a much more elegant and slightly more powerful treatment of covering theory and of Van Kampen's theorem [16, Theorem 6.7.4].

The advantages of the fundamental groupoid here and in other applications are that it can be applied also to non-connected spaces and significantly reduces the dependence on basepoints. These benefits will later be important in our main construction.

Groupoids as a tool have been heavily promoted by Ronald Brown, and we will follow his outline [16] in this section.

3.1.1 Basic Facts about Groupoids

In this section, we introduce the category of groupoids and some elementary properties of them. We also discuss the notion of homotopies between groupoids and classify groupoids up to homotopy equivalence in terms of their vertex groups. Furthermore, we give some elementary examples of groupoids.

Definition 3.1.1.

(i) A *groupoid* is a small category in which every morphism is invertible. We consider objects in a groupoid as *vertices* (in the corresponding graph) and morphisms as *elements* of the groupoid and will sometimes use notations in this spirit. In particular, if $\mathcal{G}$ is a groupoid, we will write $g \in \mathcal{G}$ to indicate that $g \in \amalg_{e,f \in \operatorname{ob} \mathcal{G}} \operatorname{Mor}_{\mathcal{G}}(e,f)$ is a morphism in $\mathcal{G}$.

(ii) A functor between groupoids is also called a *groupoid map.*

(iii) A groupoid map is called *injective/surjective* if it is injective/surjective on both objects and morphisms.

(iv) A subcategory of a groupoid $\mathcal{G}$ which is again a groupoid, is called a *subgroupoid of* $\mathcal{G}$.

(v) Suppose $f, g \colon \mathcal{G} \longrightarrow \mathcal{H}$ are groupoid maps. A natural equivalence between f and g is also called a *homotopy between* f *and* g. If such a homotopy exists, we sometimes write $f \simeq g$.

Note that by the nature of groupoids, such a homotopy h is always invertible, an inverse homotopy is given by $\overline{h} := (h_e^{-1})_{e \in \operatorname{ob} \mathcal{G}}$.

(vi) We will write Grp for the category of groupoids with groupoid maps as morphisms.

Definition 3.1.2.

(i) A groupoid $\mathcal{G}$ is called *connected*, if for each pair $i, j \in \operatorname{ob} \mathcal{G}$ there exists at least one morphism from i to j in $\mathcal{G}$ (that is, if the underlying graph of the category $\mathcal{G}$ is connected). Similarly, we get the notion of *connected components* of a groupoid.

(ii) If $\mathcal{G}$ is a groupoid, we will write $\pi_0(\mathcal{G}) \subset \operatorname{ob} \mathcal{G}$ for an (arbitrary) choice of exactly one vertex in each connected component.

(iii) If $e \in \operatorname{ob} \mathcal{G}$ is an object, we call $\mathcal{G}_e := \operatorname{Mor}_{\mathcal{G}}(e,e)$ *the vertex group of* $\mathcal{G}$ *at* e.

Example 3.1.3.

(i) A group is naturally a groupoid with exactly one object. More precisely, we can (and will) identify the category of groups with the full subcategory of the category of groupoids, having the vertex set $\{1\}$.

(ii) In this sense, the concepts (i) to (iv) in Definition 3.1.1 correspond to the obvious concepts in the group case. Two group homomorphisms $f, g\colon G \longrightarrow H$ are homotopic if and only if there exists an inner automorphism α of H, such that $\alpha \circ f = g$.

(iii) Given a family $(\mathcal{G}_i)_{i\in I}$ of groupoids, the *disjoint union of* $(\mathcal{G}_i)_{i\in I}$ is the groupoid $\amalg_{i\in I}\mathcal{G}_i$, defined by setting $\operatorname{ob}\amalg_{i\in I}\mathcal{G}_i := \amalg_{i\in I}\operatorname{ob}\mathcal{G}_i$ and

$$\forall_{k,l\in I} \quad \forall_{e\in \operatorname{ob}\mathcal{G}_k} \quad \forall_{f\in\operatorname{ob}\mathcal{G}_l} \quad \operatorname{Mor}_{\amalg_{i\in I}\mathcal{G}_i}(e,f) := \begin{cases} \operatorname{Mor}_{\mathcal{G}_k}(e,f) & \text{if } k = l \\ \emptyset & \text{else,} \end{cases}$$

together with the composition induced by the compositions of the $(\mathcal{G}_i)_{i\in I}$. In this fashion, we can view a family of groups naturally as a groupoid.

(iv) Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids. Then $\mathcal{G}\times\mathcal{H}$, i.e., the category of pairs of objects and morphisms with componentwise composition, is again a groupoid.

Example 3.1.4. For each set C there is a unique (up to canonical isomorphism) groupoid with object set C and exactly one morphism between each pair of objects, called the *simplicial groupoid with vertex set* C. For each $n \in \mathbb{N}$, we write Δ^n for the simplicial groupoid with vertex set $\{0, \ldots, n\}$.

Example 3.1.5 (Group Actions and Groupoids)**.** Let G be a group and X a set with a left G-action. We define a groupoid $G \ltimes X$, called *the action groupoid or the semi-direct product of* X *and* G, by setting:

(i) The objects of $G \ltimes X$ are given by $\operatorname{ob} G \ltimes X = X$.

(ii) For each $e, f \in X$, set $\operatorname{Mor}_{G\ltimes X}(e,f) = \{(e,g) \in X \times G \mid g \cdot e = f\}$.

(iii) Define the composition by setting for each $x \in X$ and $g, h \in G$

$$(g \cdot x, h) \circ (x, g) = (x, h \cdot g).$$

We will view groupoids as groups where the composition is only partially defined, in the sense that we can only compose two elements if the target of the first matches the source of the second. Thus it will be useful to define:

Definition 3.1.6. Let $\mathcal{G}$ be a groupoid. We define a map

$$\begin{aligned} s\colon \mathcal{G} &\longrightarrow \operatorname{ob}\mathcal{G} \\ \operatorname{Mor}_{\mathcal{G}}(e,f) \ni g &\longmapsto e \end{aligned}$$

called *source* and a map

$$\begin{aligned} t\colon \mathcal{G} &\longrightarrow \operatorname{ob}\mathcal{G} \\ \operatorname{Mor}_{\mathcal{G}}(e,f) \ni g &\longmapsto f. \end{aligned}$$

called *target*.

As the next theorem shows, up to homotopy we can actually always restrict to disjoint families of groups:

Theorem 3.1.7 (Classifying groupoids up to homotopy)**.** *Let $\mathcal{G}$ be a groupoid and $i\colon \mathcal{H} \longrightarrow \mathcal{G}$ be the inclusion of a full subgroupoid meeting each connected component of $\mathcal{G}$. Then there exists a groupoid map $p\colon \mathcal{G} \longrightarrow \mathcal{H}$, such that*

$$p \circ i = \mathrm{id}_{\mathcal{H}} \qquad \text{and} \qquad i \circ p \simeq \mathrm{id}_{\mathcal{G}}\,.$$

In particular, $\mathcal{H}$ and $\mathcal{G}$ are equivalent.

Proof. Choose a set-theoretic section $\alpha\colon \operatorname{ob}\mathcal{G} \longrightarrow \operatorname{ob}\mathcal{H}$ of $\operatorname{ob}\mathcal{H} \hookrightarrow \operatorname{ob}\mathcal{G}$ that maps vertices to vertices in the same connected component of $\mathcal{G}$. Then choose a map $e\colon \operatorname{ob}\mathcal{G} \longrightarrow \operatorname{Mor}\mathcal{G}$ such that

$$\forall_{v\in\operatorname{ob}\mathcal{G}} \quad e(v) \in \operatorname{Hom}_{\mathcal{G}}(v, \alpha(v)) \qquad \text{and} \qquad \forall_{v\in\operatorname{ob}\mathcal{H}} \quad e(v) = \mathrm{id}_v \in \operatorname{Hom}_{\mathcal{G}}(v, v).$$

Finally, define a groupoid map $p\colon \mathcal{G} \longrightarrow \mathcal{H}$ by

$$\begin{aligned}
&\forall_{v\in\operatorname{ob}\mathcal{G}} && p(v) = \alpha(v)\\
&\forall_{v,w\in\operatorname{ob}\mathcal{G}} \quad \forall_{\sigma\in\operatorname{Hom}_{\mathcal{G}}(v,w)} && p(\sigma) = e(w)\circ\sigma\circ e(v)^{-1} \in \operatorname{Hom}_{\mathcal{H}}(\alpha(v), \alpha(w)).
\end{aligned}$$

We immediately see that this map is functorial and that $p \circ i = \mathrm{id}_{\mathcal{H}}$. By construction, e is a natural equivalence between $i \circ p$ and $\mathrm{id}_{\mathcal{G}}$. □

Corollary 3.1.8.

(i) Two groupoids $\mathcal{H}$ and $\mathcal{G}$ are equivalent if and only if there is a bijection $\alpha\colon \pi_0\mathcal{G} \longrightarrow \pi_0\mathcal{H}$, such that for all $e \in \pi_0\mathcal{G}$ the vertex groups $\mathcal{G}_e$ and $\mathcal{H}_{\alpha(e)}$ are isomorphic.

(ii) In particular: Every connected non-empty groupoid is equivalent to any of its vertex groups (which coincide up to isomorphism).

Proof. Let $\varphi := (\varphi_e\colon \mathcal{G}_e \longrightarrow \mathcal{H}_{\alpha(e)})_{e\in\pi_0(\mathcal{G})}$ be a family of group isomorphisms. Then φ induces an isomorphism $\amalg_{e\in\pi_0(\mathcal{G})}\mathcal{G}_e \longrightarrow \amalg_{e\in\pi_0(\mathcal{H})}\mathcal{H}_e$ and hence

$$\mathcal{G} \simeq \amalg_{e\in\pi_0(\mathcal{G})}\mathcal{G}_e \cong \amalg_{e\in\pi_0(\mathcal{H})}\mathcal{H}_e \simeq \mathcal{H}.$$ □

The next proposition motivates the term "homotopy" for a natural equivalence:

Proposition 3.1.9. *Let $f_0, f_1\colon \mathcal{G} \longrightarrow \mathcal{H}$ be groupoid maps. Consider the two canonical inclusion maps $\mu_0, \mu_1\colon \mathcal{G} \longrightarrow \mathcal{G}\times\Delta^1$ given by*

$$\begin{aligned}
&\forall_{i\in\operatorname{ob}\mathcal{G}} && \mu_t(i) = (i, t)\\
&\forall_{g\in\operatorname{mor}\mathcal{G}} && \mu_t(g) = (g, \mathrm{id}_t)
\end{aligned}$$

for $t \in \{0, 1\}$. Then there is a one-to-one correspondence between the homotopies from f_0 to f_1 and the groupoid maps $H\colon \mathcal{G}\times\Delta^1 \longrightarrow \mathcal{H}$ satisfying $H\circ\mu_0 = f_0$ and $H\circ\mu_1 = f_1$.

Proof. Let e_{01}, e_{10} denote the two non-trivial morphisms in Δ^1. If $H\colon \mathcal{G} \times \Delta^1 \longrightarrow \mathcal{H}$ is a groupoid map satisfying $H \circ \mu_0 = f_0$ and $H \circ \mu_1 = f_1$, by setting for each $e \in \operatorname{ob} \mathcal{G}$

$$h_e := H(\mathrm{id}_e, e_{01})\colon f_0(e) \longrightarrow f_1(e)$$

we get a homotopy from f_0 to f_1, since for each pair $e, e' \in \operatorname{ob} \mathcal{G}$ and each morphism $\alpha \in \operatorname{Hom}_{\mathcal{G}}(e, e')$

$$\begin{aligned} h_{e'} \circ f_0(\alpha) &= H(\mathrm{id}_{e'}, e_{01}) \circ H(\alpha, \mathrm{id}_0) \\ &= H(\alpha, \mathrm{id}_1) \circ H(\mathrm{id}_e, e_{01}) \\ &= f_1(\alpha) \circ h_e. \end{aligned}$$

On the other hand, if h is a homotopy from f_0 to f_1, we can define a groupoid map $H\colon \mathcal{G} \times \Delta^1 \longrightarrow \mathcal{H}$ satisfying $H \circ \mu_0 = f_0$ and $H \circ \mu_1 = f_1$ by setting

$$\begin{array}{lll} \forall_{(i,t) \in \operatorname{ob} \mathcal{G} \times \Delta^1} & H(i,t) = f_t(i) & \\ \forall_{g \in \operatorname{mor} \mathcal{G}} & H(g, \mathrm{id}_0) = f_0(g) & H(g, \mathrm{id}_1) = f_1(g) \\ & H(\mathrm{id}_e, e_{01}) = h_e & H(\mathrm{id}_e, e_{10}) = h_e^{-1}. \end{array}$$

□

We give a last example that shows that for each set of vertices and each group, we can "blow up" this group to get a groupoid homotopy equivalent to the group having the given vertex set. This will be useful later when we want to consider a group G together with a family of subgroups $(A_i)_{i \in I}$ as a pair of groupoids, e.g., by considering $(G_I, \amalg_{i \in I} A_i)$.

Definition 3.1.10. Let G be a group and C a set. We define a groupoid G_C by setting

- Objects: $\operatorname{ob} G_C := C$.
- Morphisms: $\forall_{e,f \in C} \quad \operatorname{Mor}_{G_C}(e, f) := G$.

We then define composition by multiplication of elements in G, i.e., by setting for all $d, e, f \in C$

$$\begin{aligned} \operatorname{Mor}_{G_C}(e, f) \times \operatorname{Mor}_{G_C}(d, e) &\longrightarrow \operatorname{Mor}_{G_C}(d, f) \\ (g, h) &\longmapsto g \cdot h. \end{aligned}$$

Example 3.1.11. For each set C, we see that $\{1\}_C$ is the simplicial groupoid with vertex set C.

3.1.2 The Fundamental Groupoid

For us, the main examples of groupoids, besides (disjoint families of) groups, will be given by fundamental groupoids of topological spaces. These are straightforward generalisations of the fundamental group:

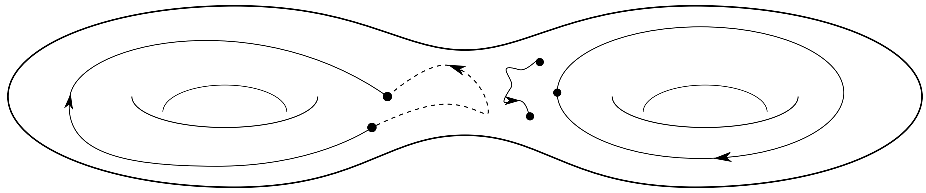

Figure 3.1: The Fundamental Groupoid – Four paths representing elements in the fundamental groupoid of the surface of genus 2.

Definition 3.1.12. Let X be a topological space and $I \subset X$ a subset. We define a groupoid $\pi_1(X, I)$ with object set I by setting

$$\forall_{i,j\in I} \quad \mathrm{Mor}_{\pi_1(X,I)}(i,j) = \{c\colon [0,1] \longrightarrow X \mid c \text{ a path from } i \text{ to } j \text{ in } X\}/\sim,$$

where $\sim$ denotes homotopy relative endpoints. We define composition via concatenation of paths. Similar to the result for the fundamental group, we see that this is indeed a well-defined groupoid, called *the fundamental groupoid of X with respect to I*.

We will also write $\pi_1(X) := \pi_1(X, X)$.

Example 3.1.13. Of course, if X is a space and $x \in X$, then $\pi_1(X, \{x\})$ is just the fundamental group of X with respect to the base point x.

Definition 3.1.14. Let (X, I) and (Y, J) be pairs of topological spaces. A continuous map $f\colon (X, I) \longrightarrow (Y, J)$ induces in the obvious way a groupoid map $\pi_1(f)\colon \pi_1(X, I) \longrightarrow \pi_1(Y, J)$:

(i) On objects we define $\pi_1(f)$ via the map

$$\begin{aligned} f|_I\colon I &\longrightarrow J \\ i &\longmapsto f(i). \end{aligned}$$

(ii) On morphisms, we set for each $i, j \in I$

$$\begin{aligned} \mathrm{Mor}_{\pi_1(X,I)}(i,j) &\longrightarrow \mathrm{Mor}_{\pi_1(Y,J)}(f(i), f(j)) \\ [\alpha] &\longmapsto [f \circ \alpha]. \end{aligned}$$

This defines a functor $\pi_1\colon \mathsf{Top}^2 \longrightarrow \mathsf{Grp}$.

Proposition 3.1.15. *Let (X, I) and (Y, J) be pairs of topological spaces. Consider maps $f, g\colon (X, I) \longrightarrow (Y, J)$. If f and g are homotopic, so are $\pi_1(f)$ and $\pi_1(g)$.*

Proof. Let $H\colon X \times [0,1] \longrightarrow Y$ be a homotopy between f and g. In particular, for all $i \in I$, the map $\sigma_i := H(i, \,\cdot\,)\colon [0,1] \longrightarrow Y$ is a path between $f(i)$ and $g(i)$. This induces a natural equivalence $([\sigma_i])_{i\in I}$ between $\pi_1(f)$ and $\pi_1(g)$: For all $i, j \in I$ and all $\alpha \in \mathrm{Mor}_{\pi_1(X,I)}(i,j)$ the following diagram commutes:

$$\begin{tikzcd} f(i) \arrow[r, "{[\sigma_i]}"] \arrow[d, "{[f \circ \alpha]}"'] & g(i) \arrow[d, "{[g \circ \alpha]}"] \\ f(j) \arrow[r, "{[\sigma_j]}"'] & g(j) \end{tikzcd}$$

Here $[\sigma_i * (f \circ \alpha)] = [(g \circ \alpha) * \sigma_j]$ holds via the given homotopy. □

3.2 Homological Algebra for Groupoids

In this section, we will discuss the algebraic setup to treat (co-)homology for groupoids, preparing the ground for our definition of bounded cohomology in the later sections. We introduce groupoid modules and generalise several algebraic constructions into this context. We then discuss the Bar resolution for groupoids, define groupoid (co-)homology with coefficients and prove some elementary properties. The Bar resolution and the corresponding definition of cohomology is a straightforward generalisation of the group case and has been extensively studied also for groupoids, for instance [71, 75].

3.2.1 Groupoid Modules

In this section we will present our definition of a module over a groupoid, generalising the group case. We will then translate basic concepts for group modules, e.g. (co-)invariants, quotients etc., to this setting. Since we are mainly interested in bounded cohomology, we will restrict ourselves to real coefficients, though we could as well work with arbitrary coefficient rings.

Definition 3.2.1 (Groupoid modules)**.** Let $\mathcal{G}$ be a groupoid. A *(left) $\mathcal{G}$-module* $V = (V_e)_{e \in \operatorname{ob} \mathcal{G}}$ consists of:

(i) A family of real vector spaces $(V_e)_{e \in \operatorname{ob} \mathcal{G}}$.

(ii) A partial action of $\mathcal{G}$ on V, i.e. for each $g \in \mathcal{G}$ a linear map

$$\begin{aligned} \rho_g \colon V_{s(g)} &\longrightarrow V_{t(g)} \\ v &\longmapsto \rho_g(v) =: gv, \end{aligned}$$

such that:

(a) For all $g, h \in \mathcal{G}$ with $s(g) = t(h)$ we have $\rho_g \circ \rho_h = \rho_{gh}$.

(b) For all $i \in \operatorname{ob} \mathcal{G}$ we have $\rho_{\mathrm{id}_i} = \mathrm{id}_{V_i}$.

Definition 3.2.2 ($\mathcal{G}$-maps)**.** Let $\mathcal{G}$ be a groupoid.

(i) Let V and W be $\mathcal{G}$-modules. An *$\mathbb{R}$-morphism between V and W* is a family $(f_e \colon V_e \longrightarrow W_e)_{e \in \operatorname{ob} \mathcal{G}}$ of $\mathbb{R}$-linear maps.

(ii) Let $f \colon V \longrightarrow W$ be an $\mathbb{R}$-morphism between $\mathcal{G}$-modules. We call f *a $\mathcal{G}$-map* or *$\mathcal{G}$-equivariant*, if for all $g \in \mathcal{G}$ we have $\rho_g^W \circ f_{s(g)} = f_{t(g)} \circ \rho_g^V$.

(iii) We write $\mathcal{G}$-Mod for the category of $\mathcal{G}$-modules and $\mathcal{G}$-maps.

Remark 3.2.3. By definition, a left $\mathcal{G}$-module is nothing else than a covariant functor $\mathcal{G} \longrightarrow \mathbb{R}$-$\mathsf{Mod}$. In this sense, a $\mathcal{G}$-map is just a natural transformation between such functors.

The functorial point of view is elegant, but the definition in terms of partial actions is more concrete and in direct analogy to the usual point of view in the group case. The latter definition will be our guide in the following sections, though we will also use the functorial definition to define some concepts more concisely.

Definition 3.2.4 (The trivial $\mathcal{G}$-module)**.** Let $\mathcal{G}$ be a groupoid. Consider the module $\mathbb{R}[\mathcal{G}] := (\mathbb{R}_e)_{e\in \operatorname{ob}\mathcal{G}} = (\mathbb{R})_{e\in \operatorname{ob}\mathcal{G}}$. We will always endow this module with the trivial $\mathcal{G}$ action given by

$$\rho_g = \mathrm{id}_{\mathbb{R}} \colon \mathbb{R}_{s(g)} \longrightarrow \mathbb{R}_{t(g)}$$

for all $g \in \mathcal{G}$.

We will see more examples in later sections. In the remainder of this section, we will translate the basic concepts of group modules into the groupoid setting, concepts that we will need later in order to do homological algebra.

Definition 3.2.5. Let $\mathcal{A}$ and $\mathcal{G}$ be groupoids and let $f\colon \mathcal{A} \longrightarrow \mathcal{G}$ be a groupoid map.

(i) Let $U\colon \mathcal{G} \longrightarrow \mathbb{R}$-$\mathsf{Mod}$ be a $\mathcal{G}$-module. We call the $\mathcal{A}$-module $f^*U := U \circ f$ *the induced $\mathcal{A}$-module structure on U.*

(ii) Let $\varphi\colon U \longrightarrow V$ be a $\mathcal{G}$-morphism. We define an $\mathcal{A}$-morphism

$$f^*\varphi\colon f^*U \longrightarrow f^*V,$$

by setting $((f^*\varphi)_e)_{e\in\operatorname{ob}\mathcal{A}} = (\varphi_{f(e)})_{e\in\operatorname{ob}\mathcal{A}}$. This defines a functor

$$f^*\colon\ \mathcal{G}\text{-}\mathsf{Mod} \longrightarrow \mathcal{A}\text{-}\mathsf{Mod}\,.$$

Lemma 3.2.6. Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids and let V be an $\mathcal{H}$-module. Let $f_0, f_1\colon \mathcal{G} \longrightarrow \mathcal{H}$ be groupoid maps and h a homotopy between f_0 and f_1. Then

$$\begin{aligned} V \circ h\colon f_0^*V &\longrightarrow f_1^*V \\ V_{f_0(e)} \ni v &\longmapsto h_e \cdot v \end{aligned}$$

is a $\mathcal{G}$-isomorphism.

We will also use a multiplicative notation and write $h\cdot v$ to denote $(V\circ h)(v)$.

Proof. By Remark 3.2.3, $V\circ h$ is a $\mathcal{G}$-map. Its inverse is given by $V\circ\overline{h}$, where $\overline{h}$ denotes the inverse homotopy to h. □

Definition 3.2.7. Let $\mathcal{G}$ be a groupoid and let V and W be $\mathcal{G}$-modules. We set

$$\operatorname{Hom}(V, W) = \{f\colon V \longrightarrow W \mid f \text{ is an } \mathbb{R}\text{-morphism of } \mathcal{G}\text{-modules}\}$$

There is a canonical $\mathcal{G}$-structure on $\operatorname{Hom}(V, W)$ given by:

(i) $\operatorname{Hom}(V, W) = (\operatorname{Hom}(V_e, W_e))_{e \in \operatorname{ob} \mathcal{G}}$.

(ii) For all $g \in \mathcal{G}$ and $f \in \operatorname{Hom}(V_{s(g)}, W_{s(g)})$ we define $g \cdot f \in \operatorname{Hom}(V_{t(g)}, W_{t(g)})$ by setting

$$\forall_{v \in V_{t(g)}} \quad (g \cdot f)(v) = g \cdot f(g^{-1} \cdot v).$$

Remark 3.2.8. Another way to see this is to view $\operatorname{Hom}(V, W)$ as the composition

$$\mathcal{G} \xrightarrow{(V, \overline{W})} (\mathbb{R}\text{-}\mathsf{Mod})^2 \xrightarrow{\text{Hom}} \mathbb{R}\text{-}\mathsf{Mod},$$

where $\overline{W}$ is the contravariant functor given by inverting morphisms in $\mathcal{G}$ and then applying W.

Remark 3.2.9. Let $\mathcal{G}$ be a groupoid. Let $f \colon V \longrightarrow W$ be a $\mathcal{G}$-map and C a $\mathcal{G}$-module. Then the dual map $\operatorname{Hom}(f, C) \colon \operatorname{Hom}(W, C) \longrightarrow \operatorname{Hom}(V, C)$ (corresponding to the family $(\operatorname{Hom}(f_e, C_e))_{e \in \operatorname{ob} \mathcal{G}}$) is again a $\mathcal{G}$-map. Alternatively, we can view $\operatorname{Hom}(f, C)$ also as the composition of the natural transformation between $(V, \overline{C})$ and $(W, \overline{C})$ induced by f and the functor Hom.

This gives rise to a contravariant functor $\operatorname{Hom}(\,\cdot\,, C) \colon \mathcal{G}\text{-}\mathsf{Mod} \longrightarrow \mathcal{G}\text{-}\mathsf{Mod}$.

Proof. The map $\operatorname{Hom}(f, C)$ is $\mathcal{G}$-equivariant since for all $g \in \mathcal{G}$, all $v \in V_{t(g)}$ and all $\alpha \in \operatorname{Hom}(W_{s(g)}, C_{s(g)})$

$$\begin{aligned}
\operatorname{Hom}(f, C)(g\alpha)(v) &= (g\alpha)(f(v)) \\
&= g(\alpha(g^{-1} f(v))) \\
&= g(\alpha(f(g^{-1} v))) \\
&= g(\alpha \circ f)(v) \\
&= g(\operatorname{Hom}(f, C)(\alpha))(v).
\end{aligned}$$ ☐

Definition 3.2.10 (Tensor products)**.** Let $\mathcal{G}$ be a groupoid and let V and W be $\mathcal{G}$-modules. We define the *tensor product of V and W over $\mathbb{R}$* to be the $\mathcal{G}$-module $V \otimes W = (V_e \otimes W_e)_{e \in \operatorname{ob} \mathcal{G}}$ with the induced $\mathcal{G}$-structure given by setting for each $g \in G$

$$\begin{aligned}
\rho_g \colon V_{s(g)} \otimes W_{s(g)} &\longrightarrow V_{t(g)} \otimes W_{t(g)} \\
v \otimes w &\longmapsto (g \cdot v) \otimes (g \cdot w).
\end{aligned}$$

In other words, $V \otimes W$ is the composition

$$\mathcal{G} \xrightarrow{(V, W)} (\mathbb{R}\text{-}\mathsf{Mod})^2 \xrightarrow{\otimes} \mathbb{R}\text{-}\mathsf{Mod}\,.$$

Definition 3.2.11 (Coinvariants and Invariants)**.** Let $\mathcal{G}$ be a groupoid and let V be a $\mathcal{G}$-module.

(i) We define the *coinvariants of V* to be the quotient module

$$V_{\mathcal{G}} = \bigoplus_{e \in \operatorname{ob} \mathcal{G}} V_e / \langle v - g \cdot v \mid g \in \mathcal{G}, v \in V_{s(g)} \rangle.$$

This construction gives rise to a functor $\mathcal{G}\text{-}\mathsf{Mod} \longrightarrow \mathbb{R}\text{-}\mathsf{Mod}$ by setting for each $\mathcal{G}$-map $\alpha\colon V \longrightarrow W$

$$\begin{aligned} \alpha_{\mathcal{G}}\colon V_{\mathcal{G}} &\longrightarrow W_{\mathcal{G}} \\ [v] &\longmapsto [\alpha(v)]. \end{aligned}$$

In other words, the coinvariants of V are just the canonical model for the colimit of $V\colon \mathcal{G} \longrightarrow \mathbb{R}\text{-}\mathsf{Mod}$, [76, Propostion 2.6.8].

(ii) We define the *invariants of* V to be the $\mathbb{R}$-subspace $V^{\mathcal{G}}$ of $\prod_{e\in \operatorname{ob}\mathcal{G}} V_e$ defined as

$$V^{\mathcal{G}} = \Big\{ v \in \prod_{e\in \operatorname{ob}\mathcal{G}} V_e \;\Big|\; \forall_{g\in\mathcal{G}} \quad g\cdot v_{s(g)} = v_{t(g)} \Big\}.$$

This construction gives rise to a functor $\mathcal{G}\text{-}\mathsf{Mod} \longrightarrow \mathbb{R}\text{-}\mathsf{Mod}$ by setting for each $\mathcal{G}$-map $\alpha\colon V \longrightarrow W$

$$\begin{aligned} \alpha^{\mathcal{G}}\colon V^{\mathcal{G}} &\longrightarrow W^{\mathcal{G}} \\ (v_e)_{e\in \operatorname{ob}\mathcal{G}} &\longmapsto (\alpha(v_e))_{e\in \operatorname{ob}\mathcal{G}}. \end{aligned}$$

In other words, the invariants of V are just the canonical model for the limit of $V\colon \mathcal{G} \longrightarrow \mathbb{R}\text{-}\mathsf{Mod}$, [76, Propostion 2.6.9].

Definition 3.2.12. Let $\mathcal{G}$ be a groupoid and V and W be $\mathcal{G}$-modules. We call

$$V \otimes_{\mathcal{G}} W := (V\otimes W)_{\mathcal{G}}$$

the *tensor product of* V *and* W *over* $\mathcal{G}$. This construction gives rise to a functor $\,\cdot \otimes_{\mathcal{G}} W\colon\ \mathcal{G}\text{-}\mathsf{Mod} \longrightarrow \mathbb{R}\text{-}\mathsf{Mod}$ by setting for each $\mathcal{G}$-map $\alpha\colon U \longrightarrow U'$

$$\begin{aligned} \alpha\otimes W\colon U\otimes_{\mathcal{G}} W &\longrightarrow U'\otimes_{\mathcal{G}} W \\ [u\otimes w] &\longmapsto [\alpha(u)\otimes w]. \end{aligned}$$

Definition 3.2.13. Let $\mathcal{G}$ be a groupoid and let V and W be $\mathcal{G}$-modules. Then we set

$$\operatorname{Hom}_{\mathcal{G}}(V,W) := \operatorname{Hom}(V,W)^{\mathcal{G}}.$$

This induces a contravariant functor $\operatorname{Hom}_{\mathcal{G}}(\,\cdot\,,W)\colon\ \mathcal{G}\text{-}\mathsf{Mod} \longrightarrow \mathbb{R}\text{-}\mathsf{Mod}$.

Definition 3.2.14 ($\mathcal{G}$-Submodules)**.** Let $\mathcal{G}$ be a groupoid and let V be a $\mathcal{G}$-module. A *$\mathcal{G}$-submodule of* V is a family $(W_e)_{e\in\operatorname{ob}\mathcal{G}}$ of real vector spaces such that

(i) For each $e \in \operatorname{ob}\mathcal{G}$ the space W_e is a subspace of V_e.

(ii) For all $g\in\mathcal{G}$

$$g\cdot W_{s(g)} \subset W_{t(g)}.$$

Then W carries a $\mathcal{G}$-module structure by considering the restricted $\mathcal{G}$-action.

Example 3.2.15. Let $f\colon V \longrightarrow W$ be a $\mathcal{G}$-map. Then its image $(f_e(V_e))_{e\in\operatorname{ob}\mathcal{G}}$ is a $\mathcal{G}$-submodule of W.

Definition 3.2.16. Let $\mathcal{G}$ be a groupoid and $\alpha\colon V \longrightarrow W$ be a $\mathcal{G}$-map. Then we call

$$\ker\alpha := (\ker\alpha_e)_{e\in\operatorname{ob}\mathcal{G}}$$

the *kernel of* α. Clearly, $\ker\alpha$ is a $\mathcal{G}$-submodule of V.

Proposition 3.2.17 (Quotients)**.** *Let $\mathcal{G}$ be a groupoid, let B be a $\mathcal{G}$-module and A a $\mathcal{G}$-submodule of B. Then there exists a canonical $\mathcal{G}$-module structure on $B/A = (B_e/A_e)_{e\in\operatorname{ob}\mathcal{G}}$ such that the family of canonical projections $\pi = (\pi_e\colon B_e \longrightarrow B_e/A_e)_{e\in\operatorname{ob}\mathcal{G}}$ is a $\mathcal{G}$-map with the following universal property: Assume that C is a $\mathcal{G}$-module and $f\colon B \longrightarrow C$ a $\mathcal{G}$-map such that $f_e(A_e) = 0$ for all $e \in \operatorname{ob}\mathcal{G}$. Then there exists a unique $\mathcal{G}$-map $\bar{f}\colon B/A \longrightarrow C$ such that $\bar{f}\circ\pi = f$.*

Proof.

- For all $g \in \mathcal{G}$, we have $g\cdot A_{s(g)} \subset A_{t(g)}$, hence there is a unique linear map $\overline{\rho_g}$ making the following diagram commutative:

$$\begin{tikzcd} B_{s(g)} \arrow[r, "\rho_g"] \arrow[d, "\pi_{s(g)}"'] & B_{t(g)} \arrow[d, "\pi_{t(g)}"] \\ B_{s(g)}/A_{s(g)} \arrow[r, "\overline{\rho_g}"'] & B_{t(g)}/A_{t(g)} \end{tikzcd}$$

  This induces a $\mathcal{G}$-structure on B/A.

- The projection $\pi\colon B \longrightarrow B/A$ is a $\mathcal{G}$-map by the definition of $\overline{\rho_g}$.

- There exists a unique family of linear maps $\bar{f} = (\bar{f}_e\colon B_e \longrightarrow B_e/A_e)_{e\in\operatorname{ob}\mathcal{G}}$ such that $\bar{f}_e\circ\pi_e = f_e$ for all $e \in \operatorname{ob}\mathcal{G}$. This map is a $\mathcal{G}$-map since for all $g \in \mathcal{G}$

$$\begin{aligned}\bar{f}_{t(g)}\circ\overline{\rho_g}\circ\pi_{s(g)} &= \bar{f}_{t(g)}\circ\pi_{t(g)}\circ\rho_g\\ &= f_{t(g)}\circ\rho_g\\ &= \rho_g\circ f_{s(g)}\\ &= \rho_g\circ\bar{f}_{s(g)}\circ\pi_{s(g)}.\end{aligned}$$

  So $\bar{f}$ is $\mathcal{G}$-equivariant. ☐

Remark 3.2.18. The notions of chain complex, chain contraction, resolution and so on, easily translate into the setting of groupoid modules.

Also, it is easy to see that $\mathcal{G}$-Mod is an Abelian category ([76, Definition A4.2]).

3.2.2 The Bar Resolution

In this section, we will define the Bar resolution for groupoids, extending the definition for the group case. We will also present a homogeneous version and show that these two resolutions are chain isomorphic. Finally, we will introduce the Bar (co-)complex with coefficients in groupoid modules, too.

Definition 3.2.19 (The Bar resolution for groupoids)**.** Let $\mathcal{G}$ be a groupoid.

(i) For each $n \in \mathbb{N}$, we set

$$P_n(\mathcal{G}) = \{(g_0, \dots, g_n) \in \mathcal{G}^{n+1} \mid \forall_{i \in \{0,\dots,n-1\}} \quad s(g_i) = t(g_{i+1})\}.$$

Equivalently, $P_n(\mathcal{G})$ is the set of all $(n+1)$-paths in $\mathcal{G}$.

(ii) For all $e \in \operatorname{ob} \mathcal{G}$ consider

$$C_n(\mathcal{G})_e = \mathbb{R}\langle\{(g_0, \dots, g_n) \in P_n(\mathcal{G}) \mid t(g_0) = e\}\rangle.$$

We define a sequence of $\mathbb{R}$-modules $(C_n(\mathcal{G}))_{n \in \mathbb{N}}$ by setting for all $n \in \mathbb{N}$

$$C_n(\mathcal{G}) = (C_n(\mathcal{G})_e)_{e \in \operatorname{ob} \mathcal{G}}$$

These modules carry a canonical $\mathcal{G}$-structure. The $\mathcal{G}$-action is then given by setting for all $g \in \mathcal{G}$

$$\begin{aligned} \rho_g \colon C_n(\mathcal{G})_{s(g)} &\longrightarrow C_n(\mathcal{G})_{t(g)} \\ (g_0, \dots, g_n) &\longmapsto (g \cdot g_0, \dots, g_n). \end{aligned}$$

(iii) For each $n \in \mathbb{N}$, we define boundary maps

$$\begin{aligned} \partial_n \colon C_n(\mathcal{G}) &\longrightarrow C_{n-1}(\mathcal{G}) \\ (g_0, \dots, g_n) &\longmapsto \sum_{i=0}^{n-1} (-1)^i (g_0, \dots, g_i \cdot g_{i+1}, \dots, g_n) \\ &\quad + (-1)^n \cdot (g_0, \dots, g_{n-1}). \end{aligned}$$

These are obviously $\mathcal{G}$-maps.

(iv) The usual calculation shows that this does indeed define a $\mathcal{G}$-chain complex $(C_n(\mathcal{G}), \partial_n)_{n \in \mathbb{N}}$.

Remark 3.2.20. Let $\mathcal{G}$ be a groupoid. Consider the canonical augmentation map

$$\begin{aligned} \varepsilon \colon C_0(\mathcal{G}) &\longrightarrow \mathbb{R}[\mathcal{G}] \\ g &\longmapsto t(g) \cdot 1. \end{aligned}$$

Then $(C_n(\mathcal{G}), \partial_n)_{n \in \mathbb{N}}$ together with ε is a $\mathcal{G}$-resolution of $\mathbb{R}[\mathcal{G}]$. An $\mathbb{R}$-chain contraction s_* is given by the $\mathbb{R}$-morphisms

$$\begin{aligned} s_{-1} \colon \mathbb{R}[\mathcal{G}] &\longrightarrow C_0(\mathcal{G}) \\ e &\longmapsto \mathrm{id}_e \end{aligned}$$

and for all $n \in \mathbb{N}$

$$\begin{aligned} s_n \colon C_n(\mathcal{G}) &\longrightarrow C_{n+1}(\mathcal{G}) \\ (g_0, \dots, g_n) &\longmapsto (\mathrm{id}_{t(g_0)}, g_0, \dots, g_n). \end{aligned}$$

Remark 3.2.21. The set $P(\mathcal{G}) := \amalg_{n\in\mathbb{N}} P_n(\mathcal{G})$ is just the underlying set of the nerve of the category $\mathcal{G}$. Therefore $(P_n(\mathcal{G}))_{n\in\mathbb{N}}$ is a simplicial set with the usual boundary maps

$$\begin{aligned}\partial_{i,n} \colon P_n(\mathcal{G}) &\longrightarrow P_{n-1}(\mathcal{G})\\ (g_0,\dots,g_n) &\longmapsto (g_0,\dots,g_i\cdot g_{i+1},\dots,g_n)\end{aligned}$$

for each $n \in \mathbb{N}_{>0}$ and $i \in \{0,\dots,n\}$ and degeneracy maps

$$\begin{aligned}s_{i,n} \colon P_n(\mathcal{G}) &\longrightarrow P_{n+1}(\mathcal{G})\\ (g_0,\dots,g_n) &\longmapsto (g_0,\dots,g_i,\mathrm{id}_{s(g_i)},g_{i+1},\dots,g_n).\end{aligned}$$

for each $n \in \mathbb{N}$ and $i \in \{0,\dots,n\}$. Taking the $\mathcal{G}$-action into account, we could view $P(\mathcal{G})$ as a functor $P(\mathcal{G}) \colon \mathcal{G} \longrightarrow \mathsf{sSet}$. The chain complex $C_*(\mathcal{G})$ can then be viewed as the usual chain complex associated to a simplicial set (but over the groupoid $\mathcal{G}$.) We refer the book of May [55] for more on simplicial sets.

As in the group case, it is sometimes helpful to use a homogeneous resolution chain isomorphic to the Bar resolution:

Definition 3.2.22 (The homogeneous Bar resolution)**.** Let $\mathcal{G}$ be a groupoid.

(i) For each $n \in \mathbb{N}$, we define a $\mathcal{G}$-module $L_n(\mathcal{G})$ by setting for each $e \in \operatorname{ob} G$

$$L_n(\mathcal{G})_e = \mathbb{R}\langle\{(g_0,\dots,g_n) \in \mathcal{G}^{n+1} \mid \forall_{i\in\{0,\dots,n\}} \quad t(g_i) = e\}\rangle.$$

and defining a $\mathcal{G}$-action by setting for each $g \in \mathcal{G}$

$$\begin{aligned}\rho_g \colon L_n(\mathcal{G})_{s(g)} &\longrightarrow L_n(\mathcal{G})_{t(g)}\\ (g_0,\dots,g_n) &\longmapsto (g\cdot g_0,\dots,g\cdot g_n).\end{aligned}$$

(ii) For each $n \in \mathbb{N}$, we define boundary maps

$$\begin{aligned}\partial_n \colon L_n(\mathcal{G}) &\longrightarrow L_{n-1}(\mathcal{G})\\ (g_0,\dots,g_n) &\longmapsto \sum_{i=0}^{n}(-1)^i(g_0,\dots,\hat{g}_i,\dots,g_n).\end{aligned}$$

These are obviously $\mathcal{G}$-maps.

(iii) The usual calculation shows that this does indeed define a $\mathcal{G}$-chain complex $(L_n(\mathcal{G}),\partial_n)_{n\in\mathbb{N}}$.

Proposition 3.2.23. *The maps*

$$\begin{aligned}C_n(\mathcal{G}) &\longrightarrow L_n(\mathcal{G})\\ (g_0,\dots,g_n) &\longmapsto (g_0,g_0\cdot g_1,\dots,g_0\cdots g_n)\end{aligned}$$

and

$$\begin{aligned}L_n(\mathcal{G}) &\longrightarrow C_n(\mathcal{G})\\ (g_0,\dots,g_n) &\longmapsto (g_0,g_0^{-1}\cdot g_1,\dots,g_{n-1}^{-1}\cdot g_n)\end{aligned}$$

are well-defined, mutually inverse $\mathcal{G}$-chain isomorphisms.

Proof. The maps are obviously mutually inverse $\mathcal{G}$-maps in each degree. They are chain maps by the same calculation as in the group case, [47, Section VI.13]. □

Remark 3.2.24. If $\mathcal{G}$ is a group, then $C_*(\mathcal{G})$ and $L_*(\mathcal{G})$ coincide with the usual definition of the (homogeneous) real Bar resolution of the group $\mathcal{G}$.

Definition 3.2.25 (Functoriality)**.** Let $f\colon \mathcal{A} \longrightarrow \mathcal{G}$ be a groupoid map. Then *the induced map*

$$\left(\begin{array}{rl} C_n(f)\colon C_n(\mathcal{A}) & \longrightarrow C_n(\mathcal{G}) \\ (a_0,\dots,a_n) & \longmapsto (f(a_0),\dots,f(a_n)) \end{array}\right)_{n\in\mathbb{N}}$$

is an $\mathcal{A}$-chain map with respect to the induced $\mathcal{A}$-structure on $C_*(\mathcal{G})$.

In order to study (bounded) (co-)homology with coefficients, it will be useful to define a domain category that encompasses both groupoids and coefficient modules. We follow Kenneth Brown [15, Section III.8] here:

Definition 3.2.26 (Domain categories for (co-)homology)**.**

(i) We define a category GrpMod by setting:

   (a) Objects in GrpMod are pairs $(\mathcal{G}, V)$, where $\mathcal{G}$ is a groupoid and V is a $\mathcal{G}$-module.

   (b) A morphism $(\mathcal{G}, V) \longrightarrow (\mathcal{H}, W)$ in GrpMod is a pair (f, φ), where $f\colon \mathcal{G} \longrightarrow \mathcal{H}$ is a groupoid map and $\varphi\colon V \longrightarrow f^*W$ is a $\mathcal{G}$-map.

   (c) Composition is defined by setting for all composable pairs of morphisms $(f,\varphi)\colon (\mathcal{G},V) \longrightarrow (\mathcal{H},W)$ and $(g,\psi)\colon (\mathcal{H},W) \longrightarrow (\mathcal{A},U)$:

$$(g,\psi)\circ(f,\varphi) := (g\circ f, (f^*\psi)\circ\varphi).$$

(ii) We define a category $\mathsf{Grp}\overline{\mathsf{Mod}}$ by setting:

   (a) Objects in $\mathsf{Grp}\overline{\mathsf{Mod}}$ are pairs $(\mathcal{G}, V)$, where $\mathcal{G}$ is a groupoid and V is a $\mathcal{G}$-module.

   (b) A morphism $(\mathcal{G}, V) \longrightarrow (\mathcal{H}, W)$ in $\mathsf{Grp}\overline{\mathsf{Mod}}$ is a pair (f, φ), where $f\colon \mathcal{G} \longrightarrow \mathcal{H}$ is a groupoid map and $\varphi\colon f^*W \longrightarrow V$ is a $\mathcal{G}$-map.

   (c) Composition is defined by setting for all composable pairs of morphisms $(f,\varphi)\colon (\mathcal{G},V) \longrightarrow (\mathcal{H},W)$ and $(g,\psi)\colon (\mathcal{H},W) \longrightarrow (\mathcal{A},U)$:

$$(g,\psi)\circ(f,\varphi) := (g\circ f, \varphi\circ(f^*\psi)).$$

Definition 3.2.27 (Bar Complex with Coefficients)**.** Let $\mathcal{G}$ be a groupoid and V a $\mathcal{G}$-module.

(i) We write

$$C_*(\mathcal{G};V) := C_*(\mathcal{G}) \otimes_{\mathcal{G}} V.$$

   Together with $\partial_* \otimes \mathrm{id}_V$, this is an $\mathbb{R}$-chain complex.

(ii) If $(f,\varphi)\colon (\mathcal{G},V) \longrightarrow (\mathcal{H},W)$ is a morphism in GrpMod, we write

$$\begin{aligned} C_*(f;\varphi) := i \circ (C_*(f) \otimes_{\mathcal{G}} \varphi)\colon C_*(\mathcal{G};V) &\longrightarrow C_*(\mathcal{H};W) \\ x \otimes v &\longmapsto f(x) \otimes \varphi(v). \end{aligned}$$

Here, let i denote the canonical $\mathbb{R}$-map

$$\begin{aligned} i\colon f^*C_*(\mathcal{H}) \otimes_{\mathcal{G}} f^*W &\longrightarrow C_*(\mathcal{H}) \otimes_{\mathcal{H}} W \\ v \otimes w &\longmapsto v \otimes w. \end{aligned}$$

This defines a functor $C_*(\,\cdot\,,\,\cdot\,)\colon \mathsf{GrpMod} \longrightarrow {}_{\mathbb{R}}\mathsf{Ch}$.

(iii) If $f\colon \mathcal{G} \longrightarrow \mathcal{H}$ is a groupoid map and W an $\mathcal{H}$-module, we also write

$$C_*(f;W) := C_*(f;\mathrm{id}_{f^*W}).$$

(iv) We write

$$C^*(\mathcal{G};V) := \mathrm{Hom}_{\mathcal{G}}(C_*(\mathcal{G}),V).$$

Together with $\delta^* := \mathrm{Hom}_{\mathcal{G}}(\partial_{*+1},V)$, this is an $\mathbb{R}$-cochain complex.

(v) If $(f,\varphi)\colon (\mathcal{G},V) \longrightarrow (\mathcal{H},W)$ is a morphism in $\mathsf{Grp}\overline{\mathsf{Mod}}$, we write $C^*(f;\varphi)$ for the map

$$\begin{aligned} \mathrm{Hom}_{\mathcal{G}}(C_*(f),\varphi) \circ j\colon C^*(\mathcal{H};W) &\longrightarrow C^*(\mathcal{G};V) \\ (w_e)_{f\in \mathrm{ob}\,\mathcal{H}} &\longmapsto (\varphi \circ w_{f(e)} \circ C_*(f))_{e\in \mathrm{ob}\,\mathcal{G}}. \end{aligned}$$

Here, let j denote the canonical $\mathbb{R}$-map

$$\begin{aligned} j\colon \mathrm{Hom}_{\mathcal{H}}(C_*(\mathcal{H}),W) &\longrightarrow \mathrm{Hom}_{\mathcal{G}}(f^*C_*(\mathcal{H}),f^*W) \\ (v_e)_{e\in \mathrm{ob}\,\mathcal{H}} &\longmapsto (v_{f(e)})_{e\in \mathrm{ob}\,\mathcal{G}}. \end{aligned}$$

This defines a contravariant functor $C^*(\,\cdot\,,\,\cdot\,)\colon \mathsf{Grp}\overline{\mathsf{Mod}} \longrightarrow {}_{\mathbb{R}}\mathsf{Ch}$.

(vi) If $f\colon \mathcal{G} \longrightarrow \mathcal{H}$ is a groupoid map and W an $\mathcal{H}$-module, we also write

$$C^*(f;W) := C^*(f;\mathrm{id}_{f^*W}).$$

3.2.3 (Co-)homology

In this section, we will introduce (co-)homology for groupoids with coefficients in groupoid modules, extending the definition in the group case. We will see that it is a groupoid homotopy invariant and can thus be calculated directly via group homology of vertex groups. For an overview of group homology, we refer to the literature [15, 47, 76].

Definition 3.2.28. Let $\mathcal{G}$ be a groupoid and V a $\mathcal{G}$-module.

(i) We call

$$H_*(\mathcal{G};V) := H_*\big(C_*(\mathcal{G};V)\big)$$

the *homology of $\mathcal{G}$ with coefficients in V*. As usual, $H_*\big(C_*(\mathcal{G};V)\big)$ denotes the homology of the $\mathbb{R}$-chain complex $C_*(\mathcal{G};V)$.

This defines a functor $H_*\colon \mathsf{GrpMod} \longrightarrow \mathbb{R}\text{-}\mathsf{Mod}_*$.

(ii) We call

$$H^*(\mathcal{G};V) := H^*\big(C^*(\mathcal{G};V)\big)$$

the *cohomology of $\mathcal{G}$ with coefficients in V*. Here, $H^*\big(C^*(\mathcal{G};V)\big)$ denotes the cohomology of the $\mathbb{R}$-cochain complex $C^*(\mathcal{G};V)$.

This defines a contravariant functor $H^*\colon \mathsf{Grp}\overline{\mathsf{Mod}} \longrightarrow \mathbb{R}\text{-}\mathsf{Mod}_*$.

Remark 3.2.29. If $\mathcal{G}$ is a group, these definitions coincide with the usual definitions of group (co-)homology.

Proposition 3.2.30. *Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids. Let $f_0, f_1\colon \mathcal{G} \longrightarrow \mathcal{H}$ be groupoid maps and let h be a homotopy from f_1 to f_0.*

(i) For each $n \in \mathbb{N}$ and each $i \in \{0,\ldots,n\}$ define a $\mathcal{G}$-map

$$\begin{aligned} s_n^i\colon C_n(\mathcal{G}) &\longrightarrow f_0^*C_{n+1}(\mathcal{H})\\ (g_0,\ldots,g_n) &\longmapsto (f_0(g_0),\ldots,f_0(g_i),h_{s(g_i)},f_1(g_{i+1}),\ldots,f_1(g_n)). \end{aligned}$$

Define for each $n \in \mathbb{N}$ a $\mathcal{G}$-map

$$s_n := \sum_{i=0}^{n}(-1)^i s_n^i.$$

Then $\partial_{n+1}\circ s_n + s_{n-1}\circ\partial_n = C_n(f_0) - h\cdot C_n(f_1)$ for each $n \in \mathbb{N}$, i.e., s_* is a $\mathcal{G}$-chain homotopy between $C_*(f_0)$ and $h\cdot C_*(f_1)$.

(ii) Let V be an $\mathcal{H}$-module. Then

$$\begin{aligned} s_*^V := i\circ(s_*\otimes_{\mathcal{G}}\mathrm{id}_{f_0^*V})\colon C_*(\mathcal{G};f_0^*V) &\longrightarrow C_{*+1}(\mathcal{H};V)\\ x\otimes v &\longmapsto s_*(x)\otimes v \end{aligned}$$

is an $\mathbb{R}$-chain homotopy between $C_*(f_0;V)$ and $C_*(f_1;V\circ\overline{h})$.

(iii) In particular,

$$H_*(f_0;V) = H_*(f_1;V\circ\overline{h}) = H_*(f_1;V)\circ H_*(\mathrm{id}_{\mathcal{G}};V\circ\overline{h});$$

and the map $H_*(\mathrm{id}_{\mathcal{G}};V\circ\overline{h})$ is an isomorphism.

(iv) Dually,

$$\begin{aligned} s_V^* := (\mathrm{Hom}_{\mathcal{G}}(s_*,\mathrm{id}_{f_0^*V}))\circ j\colon C^*(\mathcal{H},V) &\longrightarrow C^{*-1}(\mathcal{G};f_0^*V)\\ (w_e)_{e\in\mathrm{ob}\,\mathcal{H}} &\longmapsto (w_{f_0(e)}\circ s_{*,e})_{e\in\mathrm{ob}\,\mathcal{G}}. \end{aligned}$$

is an $\mathbb{R}$-cochain homotopy between $C^*(f_0;V)$ and $C^*(f_1;V\circ h)$.

(v) In particular,

$$H^*(f_0;V) = H^*(f_1;V\circ\overline{h}) = H^*(\mathrm{id}_{\mathcal{G}};V\circ\overline{h})\circ H_*(f_1;V);$$

and the map $H^*(\mathrm{id}_{\mathcal{G}};V\circ\overline{h})$ is an isomorphism.

Proof.

(i) Obviously, for each $n \in \mathbb{N}$ and $i \in \{0, \dots, n\}$ the map s_n^i is $\mathcal{G}$-equivariant. The s_n^i define a homotopy $P(\mathcal{G}) \longrightarrow P(\mathcal{H})$ of simplicial sets between $P(f_0)$ and $h \cdot P(f_1)$, see Remark 3.2.31, hence they induce a homotopy between $C_*(f_0)$ and $h \cdot C_*(f_1)$. Alternatively, this can be seen by a short calculation.

(ii) We have for each $e \in \operatorname{ob} \mathcal{G}$, each $x \in C_n(\mathcal{G})_e$ and each $v \in V_{f_0(e)}$

$$\begin{aligned}(\partial_{n+1} s_n^V + s_{n-1}^V \partial_n)(x \otimes v) &= (\partial_{n+1} s_n(x) + s_{n-1} \partial_n(x)) \otimes v. \\ &= C_n(f_0)(x) \otimes v - h_e \cdot C_n(f_1)(x) \otimes v \\ &= C_n(f_0)(x) \otimes v - C_n(f_1)(x) \otimes h_e^{-1} \cdot v \\ &= C_n(f_0; V)(x \otimes v) - C_n(f_1; V \circ \overline{h})(x \otimes v).\end{aligned}$$

(iii) The map $H_*(\mathrm{id}_{\mathcal{G}}; V \circ \overline{h})$ is an isomorphism by Lemma 3.2.6.

The parts (iv) and (v) are dual to parts (ii) and (iii). □

Remark 3.2.31. Again, it is useful to also consider a simplicial point of view. The family $(s_n^i)_{i,n}$ is directly seen to be a simplicial homotopy (over $\mathcal{G}$) between $P(f_0)$ and $h \cdot P(f_1)$.

Corollary 3.2.32. Let $f \colon G \longrightarrow \mathcal{H}$ be a homotopy equivalence between groupoids and V an $\mathcal{H}$-module. Then the induced maps

$$H_*(f; V) \colon H_*(\mathcal{G}; f^* V) \longrightarrow H_*(\mathcal{H}; V)$$

and

$$H^*(f; V) \colon H^*(\mathcal{H}; V) \longrightarrow H_*(\mathcal{G}; f^* V)$$

are isomorphisms of graded $\mathbb{R}$-modules.

Corollary 3.2.33 (Group cohomology calculates groupoid cohomology)**.** Let $\mathcal{G}$ be a connected groupoid and V a $\mathcal{G}$-module. Let $e \in \operatorname{ob} \mathcal{G}$ be a vertex and let $i_e \colon \mathcal{G}_e \longrightarrow \mathcal{G}$ be the inclusion of the corresponding vertex group. Then

$$H_*(i_e; V) \colon H_*(\mathcal{G}_e, i_e^* V) \longrightarrow H_*(\mathcal{G}; V)$$

and

$$H^*(i_e; V) \colon H^*(\mathcal{G}, V) \longrightarrow H^*(\mathcal{G}_e; i_e^* V)$$

are isomorphisms of graded $\mathbb{R}$-modules.

Proof. By Corollary 3.1.8, i_e is a homotopy equivalence and by Corollary 3.2.32, the induced maps are isomorphisms. □

Similar to the situation for topological spaces, groupoid (co-)homology is additive with respect to connected components of groupoids:

Proposition 3.2.34 (Groupoid (co-)homology and disjoint unions)**.** *Let $\mathcal{G}$ be a groupoid and V a $\mathcal{G}$-module. Let $\mathcal{G} = \amalg_{\lambda\in\Lambda}\mathcal{G}^\lambda$ be the partition of $\mathcal{G}$ into connected components. For each $\lambda \in \Lambda$, write V^λ for the $\mathcal{G}^\lambda$-module structure on V induced by the inclusion $\mathcal{G}^\lambda \hookrightarrow \mathcal{G}$. Then the family of canonical inclusions $(\mathcal{G}^\lambda \hookrightarrow \mathcal{G})_{\lambda\in\Lambda}$ induces isomorphisms*

$$\bigoplus_{\lambda\in\Lambda} H_*(\mathcal{G}^\lambda; V^\lambda) \longrightarrow H_*(\mathcal{G}; V)$$

and

$$H^*(\mathcal{G}; V) \longrightarrow \prod_{\lambda\in\Lambda} H^*(\mathcal{G}^\lambda; V^\lambda).$$

Proof. Clearly, each path in $\mathcal{G}$ is contained in a single connected component of $\mathcal{G}$. Thus we get a corresponding splitting of $C_*(\mathcal{G})$ which is compatible with the boundary maps and preserved after applying $\otimes_\mathcal{G} V$ and $\mathrm{Hom}_\mathcal{G}(\,\cdot\,, V)$. ☐

Remark 3.2.35. Combining Proposition 3.2.34 with Corollary 3.2.33, we can calculate groupoid (co-)homology completely in terms of group (co-)homology.

Example 3.2.36. Let X be a topological space. Let $\Lambda \subset X$ be a subset containing exactly one point for each connected component of X. Then the family of inclusions $(\pi_1(X, x) \hookrightarrow \pi_1(X))_{x\in\Lambda}$ induces isomorphisms of graded $\mathbb{R}$-modules

$$\bigoplus_{x\in\Lambda} H_*(\pi_1(X, x); \mathbb{R}) \longrightarrow H_*(\pi_1(X); \mathbb{R}[\pi_1(X)])$$

and

$$H^*(\pi_1(X); \mathbb{R}[\pi_1(X)]) \longrightarrow \prod_{x\in\Lambda} H^*(\pi_1(X, x); \mathbb{R}).$$

3.3 Bounded Cohomology for Groupoids

We will now define bounded cohomology and ℓ^1-homology for groupoids and derive some elementary properties.

3.3.1 Banach $\mathcal{G}$-Modules

In this section, we develop the algebraic setting to study bounded cohomology and ℓ^1-homology, introducing Banach modules over groupoids. Furthermore, we introduce bounded maps and the projective tensor product in this setting.

Definition 3.3.1 (Banach $\mathcal{G}$-Modules)**.** Let $\mathcal{G}$ be a groupoid.

(i) A *normed $\mathcal{G}$-module* is a family of normed real vector spaces $(V_e, \|\cdot\|)_{e\in\mathrm{ob}\,\mathcal{G}}$ together with a $\mathcal{G}$-structure on $(V_e)_{e\in\mathrm{ob}\,\mathcal{G}}$ such that

$$\forall_{g\in\mathcal{G}} \quad \forall_{v\in V_{s(g)}} \quad \|g\cdot v\| = \|v\|;$$

i.e., the maps $\rho_g\colon V_{s(g)} \longrightarrow V_{t(g)}$ are isometries for all $g \in \mathcal{G}$.

Equivalently, a normed $\mathcal{G}$-module is a functor $\mathcal{G} \longrightarrow \mathbb{R}\text{-}\mathsf{Mod}^1_{\|\cdot\|}$, where we denote by $\mathbb{R}\text{-}\mathsf{Mod}^1_{\|\cdot\|}$ the category of normed $\mathbb{R}$-modules together with norm-non increasing linear maps.

(ii) We call a normed $\mathcal{G}$-module V a *Banach $\mathcal{G}$-module* if for each $e \in \operatorname{ob}\mathcal{G}$, the normed $\mathbb{R}$-module $(V_e, \|\cdot\|)$ is in addition a Banach space.

(iii) Let V and W be normed $\mathcal{G}$-modules. We call a $\mathcal{G}$-map $f\colon V \longrightarrow W$ *bounded* if f_e is bounded for each $e \in \operatorname{ob}\mathcal{G}$ and the supremum $\|f\|_\infty := \sup_{e\in\operatorname{ob}\mathcal{G}} \|f_e\|_\infty$ exists. We will now always assume that *maps between normed $\mathcal{G}$-modules are bounded.*

(iv) This defines a category $\mathcal{G}\text{-}\mathsf{Mod}_{\|\cdot\|}$ of normed $\mathcal{G}$-modules and bounded $\mathcal{G}$-maps.

Example 3.3.2. Let $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ be normed $\mathcal{G}$-modules. Then the family

$$B(V, W) := (B(V_e, W_e))_{e\in\operatorname{ob}\mathcal{G}}$$

is a $\mathcal{G}$-subspace of $\operatorname{Hom}(V, W)$. Here, $B(V_e, W_e)$ denotes the space of bounded linear functions from V_e to W_e. It is a normed $\mathcal{G}$-space with respect to the family of operator norms $\|\cdot\|_\infty$ induced by the families of norms on V and W. If W is a Banach $\mathcal{G}$-module, so is $B(V, W)$. This is functorial with respect to bounded $\mathcal{G}$-maps.

Proof. For all $g \in \mathcal{G}$, all $f \in \operatorname{Hom}(V_{s(g)}, W_{s(g)})$ and all $v \in V_{t(g)}$ we have

$$\big\|(gf)(v)\big\|_W = \|gf(g^{-1}v)\|_W = \|f(g^{-1}v)\|_W.$$

Hence gf is bounded if f is bounded, so $gB(V_{s(g)}, W_{s(g)}) \subset B(V_{t(g)}, W_{t(g)})$. In this case we get by the same calculation

$$\begin{aligned}
\|gf\|_\infty &= \sup_{v\in V_{t(g)}\setminus\{0\}} \|v\|_V^{-1}\|f(g^{-1}v)\|_W \\
&\le \sup_{v\in V_{t(g)}\setminus\{0\}} \|v\|_V^{-1}\|f\|_\infty\|g^{-1}v\|_V \\
&= \|f\|_\infty.
\end{aligned}$$

And equality follows because the same is true for g^{-1}. Hence, the action is isometric. ☐

Remark 3.3.3. Alternatively, we can view $B(V, W)$ as the composition

$$\mathcal{G} \xrightarrow{(V,\overline{W})} (\mathbb{R}\text{-}\mathsf{Mod}_{\|\cdot\|})^2 \xrightarrow{B} \mathbb{R}\text{-}\mathsf{Mod}_{\|\cdot\|},$$

where $\overline{W}$ is the contravariant functor given by inverting morphisms in $\mathcal{G}$ and then applying W.

We will need a normed version of invariants of a $\mathcal{G}$-Banach module V, slightly different from the definition in Section 3.2.1 and by a slight abuse of notation, we will denote these also with $V^{\mathcal{G}}$:

Definition 3.3.4. Let $V = (V_e, \|\cdot\|_e)_{e\in \mathrm{ob}\,\mathcal{G}}$ be a normed $\mathcal{G}$-module. We call the $\mathbb{R}$-submodule of $\prod_{e\in\mathrm{ob}\,\mathcal{G}} V_e$

$$V^{\mathcal{G}} := \Big\{ v \in \prod_{e\in\mathrm{ob}\,\mathcal{G}} V_e \;\Big|\; \forall_{g\in\mathcal{G}} \quad g\cdot v_{s(g)} = v_{t(g)},\; \sup_{e\in\mathrm{ob}\,\mathcal{G}} \|v_e\|_e < \infty \Big\},$$

endowed with the norm given by setting for each $v \in V^{\mathcal{G}}$

$$\|v\| := \sup_{e\in\mathrm{ob}\,\mathcal{G}} \|v_e\|_e$$

the (normed) invariants of V.

(This can also be seen as the canonical model for the limit of $V\colon \mathcal{G} \longrightarrow \mathbb{R}\text{-}\mathsf{Mod}^1_{\|\cdot\|}$).

If $\mathcal{G}$ has only finitely many connected components, this definition coincides with Definition 3.2.11. This is functorial with respect to bounded $\mathcal{G}$-maps:

Proposition 3.3.5. *This defines a functor*

$$(\,\cdot\,)^{\mathcal{G}}\colon\ \mathcal{G}\text{-}\mathsf{Mod}_{\|\cdot\|} \longrightarrow \mathbb{R}\text{-}\mathsf{Mod}_{\|\cdot\|}\,.$$

One important example of this construction will be $B_{\mathcal{G}}(V,W) := B(V,W)^{\mathcal{G}}$.

Proposition 3.3.6 (Quotients in the Banach setting)**.** *Let B be a Banach $\mathcal{G}$-module and A a closed $\mathcal{G}$-submodule of B (i.e. $A_e \subset B_e$ closed for all $e \in \mathrm{ob}\,\mathcal{G}$). Then the family of quotient norms turns $B/A = (B_e/A_e)_{e\in\mathrm{ob}\,\mathcal{G}}$ into a Banach $\mathcal{G}$-module. We have $\|\pi\|_\infty \le 1$. Furthermore:*

Let C be a $\mathcal{G}$-module and $f\colon B \longrightarrow C$ a $\mathcal{G}$-map such that $f_e(A_e) = 0$ for all $e \in \mathrm{ob}\,\mathcal{G}$. Then there exists a unique $\mathcal{G}$-map $\bar{f}\colon B/A \longrightarrow C$ such that $\bar{f}\circ\pi = f$ and $\|\bar{f}\|_\infty = \|f\|_\infty$.

Proof. Follows from the proof of Proposition 3.2.17 and the corresponding result for regular Banach spaces. □

Remark 3.3.7. Even without $\mathcal{G}$-structures, the condition in Proposition 3.3.6 that the submodule is closed is necessary in order to get an induced norm on the quotient. This leads to the problem that Banach spaces do not form an Abelian category and methods from homological algebra for these categories cannot be applied directly to study bounded cohomology. This has prompted the development of relative homological algebra, see Section 3.4. There is however also a framework developed by Bühler to study bounded cohomology via regular homological algebra [22].

In order to introduce ℓ^1-homology of groupoids, we also need a variant of the tensor product, appropriate for the Banach-setting:

Definition 3.3.8. Let $\mathcal{G}$ be a groupoid and $(V, \|\cdot\|_V)$ and $(W, \|\cdot\|_W)$ two normed $\mathcal{G}$-modules. Then we define a norm on $V \otimes W$, called the *projective norm* by defining a norm on $V_e \otimes W_e$ for each $e \in \mathrm{ob}\,\mathcal{G}$ by

$$\begin{aligned} \|\cdot\|_\otimes\colon V_e\otimes W_e &\longrightarrow \mathbb{R} \\ x &\longmapsto \inf\Big\{ \sum_{i=1}^n \|u_i\|_V\cdot\|v_i\|_W \;\Big|\; \sum_{i=1}^n u_i\otimes v_i \text{ represents } x \text{ in } V_e\otimes W_e \Big\}. \end{aligned}$$

We call the Banach completion with respect to this norm

$$(V\overline{\otimes}W, \|\cdot\|_{\otimes}) := (\overline{V_e \otimes W_e, \|\cdot\|_{\otimes}})_{e\in \operatorname{ob}\mathcal{G}}.$$

the *projective tensor product of* V *and* W *over* $\mathbb{R}$. This is a Banach $\mathcal{G}$-module.

Furthermore, we call the normed $\mathbb{R}$-module $V \otimes_{\mathcal{G}} W := (V \otimes W)_{\mathcal{G}}$ with the induced norm the *projective tensor product of* V *and* W *over* $\mathcal{G}$.

3.3.2 Bounded Cohomology and ℓ^1-Homology

In this section, we introduce the Banach Bar (co-)complex of a groupoid with coefficients in a Banach groupoid module. We then define bounded cohomology and ℓ^1-homology for groupoids. Similarly as in Section 3.2.3, we show that these are homotopy invariants. Thus, for connected groupoids they can be calculated by regular bounded cohomology and ℓ^1-homology respectively.

Definition 3.3.9 (The Normed Bar Complex)**.** Let $\mathcal{G}$ be a groupoid.

(i) For each $n \in \mathbb{N}$, we can put a norm on $C_n(\mathcal{G})$ by endowing $C_n(\mathcal{G})_e$ with the ℓ^1-norm with respect to $P_n(\mathcal{G})_e$ for each $e \in \operatorname{ob}\mathcal{G}$:

$$\begin{aligned} \|\cdot\|_1 \colon C_n(\mathcal{G})_e &\longrightarrow \mathbb{R} \\ \sum_{\sigma\in P_n(\mathcal{G})_e} \lambda_\sigma \cdot \sigma &\longmapsto \sum_{\sigma\in P_n(\mathcal{G})_e} |\lambda_\sigma|. \end{aligned}$$

In this way, $(C_n(\mathcal{G})_e, \|\cdot\|)_{e\in\operatorname{ob}\mathcal{G}}$ becomes a normed $\mathcal{G}$-module.

(ii) By definition, the boundary maps are bounded $\mathcal{G}$-maps and for each $n \in \mathbb{N}$ we have $\|\partial_n\|_\infty \le n+1$, so $(C_n(\mathcal{G}), \|\cdot\|_1, \partial_n)$ is a normed $\mathcal{G}$-chain complex.

(iii) If $f\colon \mathcal{G} \longrightarrow \mathcal{H}$ is a groupoid map, the induced map $C_n(f)$ is bounded for each $n \in \mathbb{N}$ and satisfies

$$\|C_n(f)\|_\infty \le 1,$$

since $C_n(f)$ maps simplices to simplices.

We define domain categories GrpBan and $\mathsf{Grp}\overline{\mathsf{Ban}}$ for ℓ^1-homology and bounded cohomology, completely analogously to GrpMod and $\mathsf{Grp}\overline{\mathsf{Mod}}$ by simply replacing groupoid modules with Banach groupoid modules:

Definition 3.3.10 (Domain Categories for Bounded Cohomology)**.**

(i) We define a category GrpBan by setting

  (a) Objects in GrpBan are pairs $(\mathcal{G}, V)$, where $\mathcal{G}$ is a groupoid and V is a Banach $\mathcal{G}$-module.

  (b) A morphism $(\mathcal{G}, V) \longrightarrow (\mathcal{H}, W)$ in GrpBan is a pair (f, φ), where $f\colon \mathcal{G} \longrightarrow \mathcal{H}$ is a groupoid map and $\varphi\colon V \longrightarrow f^*W$ is a bounded $\mathcal{G}$-map.

  (c) Composition is defined by setting for all composable pairs of morphisms $(f, \varphi)\colon (\mathcal{G}, V) \longrightarrow (\mathcal{H}, W)$ and $(g, \psi)\colon (\mathcal{H}, W) \longrightarrow (\mathcal{A}, U)$:

$$(g, \psi) \circ (f, \varphi) := (g \circ f, (f^*\psi) \circ \varphi).$$

(ii) We define a category $\mathsf{Grp}\overline{\mathsf{Ban}}$ by setting

(a) Objects in $\mathsf{Grp}\overline{\mathsf{Ban}}$ are pairs $(\mathcal{G}, V)$, where $\mathcal{G}$ is a groupoid and V is a Banach $\mathcal{G}$-module.

(b) A morphism $(\mathcal{G}, V) \longrightarrow (\mathcal{H}, W)$ in $\mathsf{Grp}\overline{\mathsf{Ban}}$ is a pair (f, φ), where $f\colon \mathcal{G} \longrightarrow \mathcal{H}$ is a groupoid map and $\varphi\colon f^*W \longrightarrow V$ is a bounded $\mathcal{G}$-map.

(c) Composition is defined by setting for all composable pairs of morphisms $(f, \varphi)\colon (\mathcal{G}, V) \longrightarrow (\mathcal{H}, W)$ and $(g, \psi)\colon (\mathcal{H}, W) \longrightarrow (\mathcal{A}, U)$:

$$(g, \psi) \circ (f, \varphi) := (g \circ f, \varphi \circ (f^*\psi)).$$

Definition 3.3.11 (The Banach Bar Complex with Coefficients)**.** Let $\mathcal{G}$ be a groupoid and V a Banach $\mathcal{G}$-module.

(i) We write

$$C_*^{\ell^1}(\mathcal{G}; V) := C_*(G)\overline{\otimes}_{\mathcal{G}} V.$$

Together with the projective tensor product norm, this is a normed $\mathbb{R}$-chain complex.

(ii) If $(f, \varphi)\colon (\mathcal{G}, V) \longrightarrow (\mathcal{H}, W)$ is a morphism in GrpBan, the map

$$C_*(f; \varphi)\colon C_*(\mathcal{G}; V) \longrightarrow C_*(\mathcal{H}; W)$$

is bounded with respect to the tensor product norms, hence induces a bounded $\mathbb{R}$-map

$$C_*^{\ell^1}(f; \varphi)\colon C_*^{\ell^1}(\mathcal{G}; V) \longrightarrow C_*^{\ell^1}(\mathcal{H}; W)$$

This defines a functor $C_*^{\ell^1}\colon \mathsf{GrpBan} \longrightarrow {}_{\mathbb{R}}\mathsf{Ch}^{\|\cdot\|}$.

(iii) We write

$$C_b^*(\mathcal{G}; V) := B_{\mathcal{G}}(C_*(G), V).$$

Together with $\|\cdot\|_\infty$, this is a normed $\mathbb{R}$-cochain complex.

(iv) If $(f, \varphi)\colon (\mathcal{G}, V) \longrightarrow (\mathcal{H}, W)$ is a morphism in $\mathsf{Grp}\overline{\mathsf{Ban}}$, we write $C_b^*(f; \varphi)$ for the map

$$\begin{aligned} C_b^*(\mathcal{H}; W) &\longrightarrow C_b^*(\mathcal{G}; V) \\ (w_e)_{f \in \operatorname{ob}\mathcal{H}} &\longmapsto (\varphi \circ w_{f(e)} \circ C_*(f))_{e \in \operatorname{ob}\mathcal{G}}. \end{aligned}$$

This defines a contravariant functor $C_b^*\colon \mathsf{Grp}\overline{\mathsf{Ban}} \longrightarrow {}_{\mathbb{R}}\mathsf{Ch}^{\|\cdot\|}$.

Definition 3.3.12. Let $\mathcal{G}$ be a groupoid, V a Banach $\mathcal{G}$-module.

(i) We call the homology

$$H_*^{\ell^1}(\mathcal{G}; V) := H_*(C_*^{\ell^1}(\mathcal{G}; V))$$

together with the induced semi-norm on $H_*^{\ell^1}(\mathcal{G}; V)$ the *ℓ^1-homology of $\mathcal{G}$ with coefficients in V*.

This defines a functor $H_*^{\ell^1}\colon \mathsf{GrpBan} \longrightarrow \mathbb{R}\text{-}\mathsf{Mod}_*^{\|\cdot\|}$

(ii) We call the cohomology

$$H_b^*(\mathcal{G};V) := H^*(B_{\mathcal{G}}(C_*(\mathcal{G}),V)),$$

together with the induced semi-norm on $H_b^*(\mathcal{G};V)$, the *bounded cohomology of $\mathcal{G}$ with coefficients in V.*

This defines a contravariant functor $H_b^* \colon \mathsf{Grp}\overline{\mathsf{Ban}} \longrightarrow \mathbb{R}\text{-}\mathsf{Mod}_*^{\|\cdot\|}$.

Remark 3.3.13. As before, if $\mathcal{G}$ is a group, our definition of ℓ^1-homology and bounded cohomology coincides with the usual one.

Since the chain homotopies we have considered in Proposition 3.2.30 are bounded, we get the following analogue of the homotopy invariance of groupoid homology:

Proposition 3.3.14. *Let $\mathcal{G}$ and $\mathcal{H}$ be groupoids. Let $f_0, f_1 \colon \mathcal{G} \longrightarrow \mathcal{H}$ be groupoid maps and let h be a homotopy from f_1 to f_0.*

(i) For each $n \in \mathbb{N}$ the map s_n defined in Proposition 3.2.30 is a bounded $\mathcal{G}$-map and $\|s_n\|_\infty \leq n+1$.

(ii) Let V be an $\mathcal{H}$-module. For each $n \in \mathbb{N}$, the chain homotopy s_n^V is bounded and hence induces an $\mathbb{R}$-chain homotopy between the maps $C_*^{\ell^1}(f_0;V)$ and $C_*^{\ell^1}(f_1; V \circ \overline{h})$.

(iii) In particular,

$$H_*^{\ell^1}(f_0;V) = H_*^{\ell^1}(f_1;V) \circ H_*^{\ell^1}(\mathrm{id}_{\mathcal{G}}; V \circ \overline{h});$$

and the map $H_*^{\ell^1}(\mathrm{id}_{\mathcal{G}}; V \circ \overline{h})$ is an isometric isomorphism.

(iv) Dually, s_V^* induces an $\mathbb{R}$-cochain homotopy between the maps $C_b^*(f_0;V)$ and $C_b^*(f_1; V \circ h)$.

(v) In particular,

$$H_b^*(f_0;V) = H_b^*(\mathrm{id}_{\mathcal{G}}; V \circ \overline{h}) \circ H_b^*(f_1;V);$$

and the map $H_b^*(\mathrm{id}_{\mathcal{G}}; V \circ \overline{h})$ is an isometric isomorphism.

Proof. The only thing left to show is that $H_*^{\ell^1}(\mathrm{id}_{\mathcal{G}}; V \circ \overline{h})$ and $H_b^*(\mathrm{id}_{\mathcal{G}}; V \circ \overline{h})$ are isometric. But

$$V \circ \overline{h} \colon f_0^* V \longrightarrow f_1^* V$$
$$V_{f_0(e)} \ni v \longmapsto \overline{h}_e \cdot v$$

is isometric since the $\mathcal{H}$-action on V is isometric, hence all the induced maps are also isometric. □

Corollary 3.3.15. Let $f \colon \mathcal{G} \longrightarrow \mathcal{H}$ be an equivalence between groupoids and V a Banach $\mathcal{H}$-module. Then the induced maps

$$H_*^{\ell^1}(f;V) \colon H_*^{\ell^1}(\mathcal{G}; f^*V) \longrightarrow H_*^{\ell^1}(\mathcal{H};V)$$

and

$$H_b^*(f;V) \colon H_b^*(\mathcal{H};V) \longrightarrow H_b^*(\mathcal{G}; f^*V)$$

are isometric isomorphisms of semi-normed graded $\mathbb{R}$-modules.

Corollary 3.3.16 (Bounded group cohomology calculates bounded groupoid cohomology)**.** Let $\mathcal{G}$ be a connected groupoid and V a Banach $\mathcal{G}$-module. Let $e \in \operatorname{ob}\mathcal{G}$ be a vertex and $i_e \colon \mathcal{G}_e \longrightarrow \mathcal{G}$ the inclusion of the corresponding vertex group. Then

$$H_*^{\ell^1}(i_e, V)\colon H_*^{\ell^1}(\mathcal{G}_e, i_e^* V) \longrightarrow H_*^{\ell^1}(\mathcal{G}; V)$$

and

$$H_b^*(i_e, V)\colon H_b^*(\mathcal{G}, V) \longrightarrow H_b^*(\mathcal{G}_e; i_e^* V)$$

are isometric isomorphisms of normed graded $\mathbb{R}$-modules.

Proof. By Corollary 3.1.8, i_e is a homotopy equivalence and by Corollary 3.3.15, the induced maps are isometric isomorphisms. Thus the result follows form Remark 3.3.13. □

Definition 3.3.17. Let $(V_i, \|\cdot\|_i)_{i\in I}$ be a *finite* family of (semi-)normed $\mathbb{R}$-modules.

(i) We define *the direct sum (semi-)norm* on $\bigoplus_{i\in I} V_i$ by setting

$$\begin{aligned} \bigoplus_{i\in I} V_i &\longrightarrow \mathbb{R} \\ \sum_{i\in I} v_i &\longmapsto \sum_{i\in I} \|v_i\|_i. \end{aligned}$$

(ii) We define *the product (semi-)norm* on $\prod_{i\in I} V_i$ by setting

$$\begin{aligned} \prod_{i\in I} V_i &\longrightarrow \mathbb{R} \\ (v_i)_{i\in I} &\longmapsto \max\{\|v_i\|_i \mid i \in I\}. \end{aligned}$$

Proposition 3.3.18 (Bounded groupoid cohomology and disjoint unions)**.** *Let $\mathcal{G}$ be a groupoid having finitely many connected components and V a Banach $\mathcal{G}$-module. Let $\mathcal{G} = \amalg_{\lambda\in\Lambda}\mathcal{G}^\lambda$ be the partition of $\mathcal{G}$ into connected components. For each $\lambda \in \Lambda$, write V^λ for the $\mathcal{G}$-module structure on V induced by the inclusion $\mathcal{G}^\lambda \hookrightarrow \mathcal{G}$. Then the family of inclusions $(\mathcal{G}^\lambda \hookrightarrow \mathcal{G})_{\lambda\in\Lambda}$ induces isometric isomorphisms*

$$\bigoplus_{\lambda\in\Lambda} H_*^{\ell^1}(\mathcal{G}^\lambda; V^\lambda) \longrightarrow H_*^{\ell^1}(\mathcal{G}; V)$$

with respect to the direct sum semi-norms and

$$H_b^*(\mathcal{G}; V) \longrightarrow \prod_{\lambda\in\Lambda} H_b^*(\mathcal{G}^\lambda; V^\lambda)$$

with respect to the product semi-norm.

Proof. We see directly that the splitting of $C_*(\mathcal{G})$ with respect to the connected components is preserved after applying $\overline{\otimes}_{\mathcal{G}} V$ and $B_{\mathcal{G}}(\,\cdot\,, V)$ and that the norm is exactly the direct sum norm or respectively the product norm. □

3.4 Relative Homological Algebra for Groupoids

In this section, we develop the relative homological algebra necessary to study resolutions that can calculate bounded cohomology and ℓ^1-homology of groupoids respectively, analogously to the group case. We introduce the notations of relatively injective and relatively projective groupoid modules and define strong resolutions for Banach groupoid modules. Then, we show that $B(C_*(\mathcal{G}), V)$ is a strong relatively injective resolution for each Banach $\mathcal{G}$-module V. Next, we prove the fundamental lemma for relative homological algebra in our setting, implying in particular that strong relatively injective resolutions are unique up to bounded $\mathcal{G}$-cochain equivalence. Thus these resolutions can be used to calculate bounded cohomology up to isomorphism, and we show that the semi-norm on bounded cohomology can be seen to be the infimum over all semi-norms induced by strong relatively injective resolutions. Finally, we also discuss the dual results for ℓ^1-homology.

Definition 3.4.1. Let $\mathcal{G}$ be a groupoid.

(i) Let V and W be Banach $\mathcal{G}$-modules. A $\mathcal{G}$-map $i\colon V \longrightarrow W$ is called *relatively injective* if there exists a (not necessarily $\mathcal{G}$-equivariant) $\mathbb{R}$-morphism $\sigma\colon W \longrightarrow V$ such that $\sigma \circ i = \mathrm{id}_V$ and $\|\sigma\|_\infty \leq 1$.

(ii) A $\mathcal{G}$-module I is called *relatively injective* if for each relatively injective $\mathcal{G}$-map $i\colon V \longrightarrow W$ between Banach $\mathcal{G}$-modules and each $\mathcal{G}$-map $\alpha\colon V \longrightarrow I$ there is a $\mathcal{G}$-map $\beta\colon W \longrightarrow I$, such that $\beta \circ i = \alpha$ and $\|\beta\|_\infty \leq \|\alpha\|_\infty$.

Dually, we also define relatively projective modules:

Definition 3.4.2. Let $\mathcal{G}$ be a groupoid.

(i) Let V and W be Banach $\mathcal{G}$-modules. A $\mathcal{G}$-map $p\colon V \longrightarrow W$ is called *relatively projective* if there exists a (not necessarily $\mathcal{G}$-equivariant) $\mathbb{R}$-morphism $\sigma\colon W \longrightarrow V$ such that $p \circ \sigma = \mathrm{id}_W$ and $\|\sigma\|_\infty \leq 1$.

(ii) A $\mathcal{G}$-module P is called *relatively projective* if for each relatively projective $\mathcal{G}$-map $p\colon V \longrightarrow W$ between Banach $\mathcal{G}$-modules and each $\mathcal{G}$-map $\alpha\colon P \longrightarrow W$ there is a $\mathcal{G}$-map $\beta\colon P \longrightarrow V$, such that $p \circ \beta = \alpha$ and $\|\beta\|_\infty \leq \|\alpha\|_\infty$.

Figure 3.1 illustrates the mapping problems for relatively injective and projective modules.

Proposition 3.4.3. *Let U be a Banach $\mathcal{G}$-module and $n \in \mathbb{N}$. Then the Banach $\mathcal{G}$-module $B(C_n(\mathcal{G}), U)$ is relatively injective.*

The proof is basically the same as in the case of $\mathcal{G}$ being a group:

Proof. Let V and W be Banach $\mathcal{G}$-modules and $i\colon V \longrightarrow W$ be a relatively injective $\mathcal{G}$-map. Let $\sigma\colon W \longrightarrow V$ be as in Definition 3.4.1(i). Furthermore, let $\alpha\colon V \longrightarrow B(C_n(\mathcal{G}), U)$ be a bounded $\mathcal{G}$-map. We define a family of linear maps $(\beta_e\colon W_e \longrightarrow B(C_n(\mathcal{G})_e, U_e))_{e \in \operatorname{ob}\mathcal{G}}$ by setting for each $e \in \operatorname{ob}\mathcal{G}$

$$\begin{aligned}\beta_e\colon W_e &\longrightarrow B(C_n(\mathcal{G})_e, U_e)\\ w &\longmapsto \big((g_0,\dots,g_n) \longmapsto \alpha_e(g_0 \cdot \sigma_{s(g_0)}(g_0^{-1} \cdot w))(g_0,\dots,g_n)\big)\end{aligned}$$

$$
\begin{tikzcd}
& W \arrow[dd, bend left, "\sigma"] \\
P \arrow[ur, "\alpha"] \arrow[dr, dashed, "\beta"'] & \\
& V \arrow[uu, "p"]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
V \arrow[dd, "i"] \arrow[dr, "\alpha"] & \\
& I \\
W \arrow[uu, bend left, "\sigma"] \arrow[ur, dashed, "\beta"'] &
\end{tikzcd}
$$

Figure 3.2: Mapping problem for relatively projective and relatively injective modules.

- For each $e \in \operatorname{ob}\mathcal{G}$ the map β_e is bounded and $\|\beta_e\|_\infty \leq \|\alpha_e\|_\infty$: We have for each $g_0 \in \mathcal{G}$ and each $w \in W_{t(g_0)}$

$$
\begin{aligned}
\|\alpha_{t(g_0)}(g_0 \cdot \sigma_{s(g_0)}(g_0^{-1} \cdot w))\|_\infty &\leq \|\alpha_{t(g_0)}\|_\infty \cdot \|g_0 \cdot \sigma_{s(g_0)}(g_0^{-1} \cdot w))\|_\infty \\
&= \|\alpha_{t(g_0)}\|_\infty \cdot \|\sigma_{s(g_0)}(g_0^{-1} \cdot w))\|_\infty \\
&\leq \|\alpha_{t(g_0)}\|_\infty \cdot \|\sigma_{s(g_0)}\|_\infty \cdot \|(g_0^{-1} \cdot w))\| \\
&\leq \|\alpha_{t(g_0)}\|_\infty \cdot \|w\|.
\end{aligned}
$$

- The map β is $\mathcal{G}$-equivariant: We have for all $g \in \mathcal{G}$, $w \in W_{s(g)}$ and all $(g_0, \ldots, g_n)$ in $P_n(\mathcal{G})$ with $t(g_0) = t(g)$

$$
\begin{aligned}
&\beta_{t(g)}(g \cdot w)(g_0, \ldots, g_n) = \alpha_{t(g)}(g_0 \cdot \sigma_{s(g_0)}(g_0^{-1} \cdot g \cdot w))(g_0, \ldots, g_n) \\
&\quad = (g \cdot \alpha_{s(g)}(g^{-1} \cdot g_0 \cdot \sigma_{s(g_0)}((g^{-1} \cdot g_0)^{-1} \cdot w))(g_0, \ldots, g_n) \\
&\quad = g \cdot (\alpha_{s(g)}(g^{-1} \cdot g_0 \cdot \sigma_{s(g_0)}((g^{-1} \cdot g_0)^{-1} \cdot w))(g^{-1} \cdot g_0, \ldots, g_n) \\
&\quad = g \cdot (\beta_{s(g)}(w)(g^{-1} \cdot g_0, \ldots, g_n)) \\
&\quad = (g \cdot \beta_{s(g)}(w))(g_0, \ldots, g_n).
\end{aligned}
$$

  Hence $\beta_{t(g)}(g \cdot w) = g \cdot \beta_{s(g)}(w)$.

- Since we have $\|\beta_e\|_\infty \leq \|\alpha_e\|_\infty$ for all $e \in \operatorname{ob}\mathcal{G}$, the map β is bounded and $\|\beta\|_\infty \leq \|\alpha\|_\infty$.

- For all $(g_0, \ldots, g_n) \in P_n(\mathcal{G})$ and $w \in W_{t(g_0)}$ we have

$$
\begin{aligned}
(\beta_{t(g_0)} \circ i_{t(g_0)})(w)(g_0, \ldots g_n) &= \alpha_{t(g_0)}(g_0 \cdot \sigma_{s(g_0)}(g_0^{-1} \cdot i_{t(g_0)}(w))(g_0, \ldots, g_n) \\
&= \alpha_{t(g_0)}(g_0 \cdot \sigma_{s(g_0)}(i_{s(g_0)}(g_0^{-1} \cdot w))(g_0, \ldots, g_n) \\
&= \alpha_{t(g_0)}(g_0 \cdot (g_0^{-1} \cdot w))(g_0, \ldots, g_n) \\
&= \alpha_{t(g_0)}(w)(g_0, \ldots, g_n).
\end{aligned}
$$

  Hence $\alpha = \beta \circ i$. □

Dually, we also have:

Proposition 3.4.4. *Let $\mathcal{G}$ be a groupoid and U a Banach $\mathcal{G}$-module. Then for all $n \in \mathbb{N}$, $C_n(\mathcal{G}) \overline{\otimes} U$ is relatively projective.*

Proof. Let $p\colon V \longrightarrow W$ be a relatively projective $\mathcal{G}$-map between Banach $\mathcal{G}$-modules and $\alpha\colon C_n(\mathcal{G})\overline{\otimes}U \longrightarrow W$ a $\mathcal{G}$-map. Let $\sigma\colon W \longrightarrow V$ be as in Definition 3.4.2 (i). Define a $\mathcal{G}$-map

$$\begin{aligned}\beta\colon C_n(\mathcal{G})\overline{\otimes}U &\longrightarrow V\\ (g_0,\dots,g_n)\otimes u &\longmapsto g_0\cdot\sigma\big(\alpha((\mathrm{id}_{s(g_0)},g_1,\dots,g_n)\otimes g_0^{-1}\cdot u)\big).\end{aligned}$$

A calculation similar to the one for the relative injective case shows that this map indeed has the wanted properties. □

Definition 3.4.5 (Strong Resolutions)**.** Let (C_*,∂_*) be a Banach $\mathcal{G}$-chain complex, V a Banach $\mathcal{G}$-module and $\varepsilon\colon C_0 \longrightarrow V$ a $\mathcal{G}$-augmentation map. We call $(C_*,\partial_*,\varepsilon)$ *strong* or *a strong resolution for* V if there exists a norm non-increasing chain contraction, i.e., a family

$$\begin{aligned}(s_n\colon C_n &\longrightarrow C_{n+1})_{n\in\mathbb{N}}\\ s_{-1}\colon V &\longrightarrow C_0\end{aligned}$$

of (not necessarily $\mathcal{G}$-equivariant) $\mathbb{R}$-morphisms between $\mathcal{G}$-modules such that

(i) For all $n\in\mathbb{N}\cup\{-1\}$ we have $\|s_n\|_\infty\leq 1$.

(ii) The family $(s_n)_{n\in\mathbb{N}}$ is a chain contraction of the augmented chain complex, i.e., for all $n\in\mathbb{N}_{>0}$

$$s_{n-1}\circ\partial_n+\partial_{n+1}\circ s_n=\mathrm{id}_{C_n},$$

and

$$\begin{aligned}s_{-1}\circ\varepsilon+\partial_1\circ s_0&=\mathrm{id}_{C_0}\\ \varepsilon\circ s_{-1}&=\mathrm{id}_V\,.\end{aligned}$$

Dually, we also define *strong resolutions for cochain complexes.*

Example 3.4.6. Let $\mathcal{G}$ be a groupoid and V a Banach $\mathcal{G}$-module. Recall from Remark 3.2.20 the definition of the chain contraction $s_*\colon C_*(\mathcal{G})\longrightarrow C_{*+1}(\mathcal{G})$:

$$\begin{aligned}s_{-1}\colon\overline{\mathbb{R}}&\longrightarrow C_0(\mathcal{G})\\ e&\longmapsto\mathrm{id}_e\end{aligned}$$

and for all $n\in\mathbb{N}$

$$\begin{aligned}s_n\colon C_n(\mathcal{G})&\longrightarrow C_{n+1}(\mathcal{G})\\ (g_0,\dots,g_n)&\longmapsto(\mathrm{id}_{t(g_0)},g_0,\dots,g_n).\end{aligned}$$

(i) Consider the Banach $\mathcal{G}$-chain complex $C_*(G)\overline{\otimes}V$ together with the augmentation map

$$\begin{aligned}\varepsilon\colon C_0^{\ell^1}(\mathcal{G};V)&\longrightarrow V\\ g\otimes v&\longmapsto v.\end{aligned}$$

The family s_* induces a norm-non increasing $\mathbb{R}$-chain contraction $s_*^{\ell^1}$ of $(C_*(G)\overline{\otimes}V, \varepsilon)$ given by

$$\begin{aligned} s_*^{\ell^1} : C_*^{\ell^1}(\mathcal{G}; V) &\longrightarrow C_{*+1}^{\ell^1}(\mathcal{G}; V) \\ x \otimes v &\longmapsto s_*(x) \otimes v \end{aligned}$$

and

$$\begin{aligned} s_{-1}^{\ell^1} \colon V &\longrightarrow C_0^{\ell^1}(\mathcal{G}; V) \\ V_e \ni v &\longmapsto \mathrm{id}_e \otimes v. \end{aligned}$$

In particular, $(C_*(\mathcal{G})\overline{\otimes}V, \varepsilon)$ is a strong (homological) $\mathcal{G}$-resolution of V.

(ii) Consider the Banach $\mathcal{G}$-cochain complex $B(C_*(\mathcal{G}), V)$ together with the augmentation map $\varepsilon\colon B(C_0(\mathcal{G}), V) \longrightarrow V$ given by

$$\begin{aligned} \varepsilon\colon V &\longrightarrow B(C_0(\mathcal{G}), V) \\ v &\longmapsto (g \longmapsto v). \end{aligned}$$

Then, the family s_* induces a norm-non increasing $\mathbb{R}$-cochain contraction s_b^* of $(B(C_*(\mathcal{G}), V), \varepsilon)$ given by

$$\begin{aligned} s_b^*\colon B(C_*(\mathcal{G}), V) &\longrightarrow B(C_{*-1}(\mathcal{G}), V) \\ \varphi &\longmapsto \varphi \circ s_* \end{aligned}$$

and setting for each $e \in \operatorname{ob}\mathcal{G}$

$$\begin{aligned} s_{b,e}^0\colon B(C_0(\mathcal{G})_e, V_e) &\longrightarrow V_e \\ \varphi &\longmapsto \varphi(\mathrm{id}_e). \end{aligned}$$

In particular, $(B(C_*(\mathcal{G}), V), \varepsilon)$ is a strong (cohomological) resolution of V.

Proposition 3.4.7 (Fundamental lemma I)**.** *Let $(I^n, \delta_I^n)_{n\in\mathbb{N}}$ be a relatively injective Banach $\mathcal{G}$-cochain complex and $\varepsilon\colon W \longrightarrow I^0$ a $\mathcal{G}$-augmentation map. Let $((C^n, \delta_C^n)_{n\in\mathbb{N}}, \nu\colon V \longrightarrow C^0)$ be a strong $\mathcal{G}$-resolution of a Banach $\mathcal{G}$-module V.*

Let $f\colon V \longrightarrow W$ be a $\mathcal{G}$-morphism. Then there exists an up to bounded $\mathcal{G}$-cochain homotopy unique extension of f to a bounded $\mathcal{G}$-cochain map between the resolution (C^n, δ_C^n, ν) and the augmented cochain complex $(I^n, \delta_I^n, \varepsilon)$.

Proof. The proof is basically the same lifting argument as for all homological "fundamental lemmas", for instance [62, Lemma 7.2.4].

Let $((s^n\colon C^n \longrightarrow C^{n-1})_{n\in\mathbb{N}_{>0}}, s^0\colon C^0 \longrightarrow V)$ be a norm non-increasing cochain contraction of the augmented cochain complex. We construct the sequence $(f^n)_{n\in\mathbb{N}}$ by induction.

The augmentation map $\varepsilon\colon V \longrightarrow C^0$ is relatively injective ($s^0 \circ \varepsilon = \mathrm{id}_V$). For this reason, because I^0 is relatively injective, there exists a bounded $\mathcal{G}$-lift $f^0\colon C^0 \longrightarrow I^0$ of $\nu \circ f$ such that $f^0 \circ \varepsilon = \nu \circ f$.

$$\begin{tikzcd} C^0 \arrow[r, dashed, "f^0"] \arrow[d, shift left, "s^0"] & I^0 \\ V \arrow[u, shift left, "\varepsilon"] \arrow[r, "f"] & W \arrow[u, "\nu"'] \end{tikzcd}$$

Assume now that for an $n \in \mathbb{N}$ bounded $\mathcal{G}$-maps $f^0, \dots, f^n$ have been constructed, compatible with the boundary maps. The $\mathcal{G}$-submodules $\ker \delta_I^n \subset I^n$ and $\ker \delta_C^n \subset C^n$ are closed. Since $f^n \circ \delta_C^{n-1} = \delta_I^{n-1} \circ f^{n-1}$, and because $\operatorname{im} \delta_I^{n-1} \subset \ker \delta_I^n$ and $\operatorname{im} \delta_C^{n-1} \subset \ker \delta_C^n$, the map f^n induces a $\mathcal{G}$-map $\overline{f^n} \colon C^n / \ker \delta_C^n \longrightarrow I^n / \ker \delta_I^n$ such that $\overline{f^n} \circ \pi_{\ker \delta_C^n} = \pi_{\ker \delta_I^n} \circ f^n$, where $\pi_{\ker \delta_I^n}$ and $\pi_{\ker \delta_I^n}$ are the canonical projections:

$$\begin{tikzcd}
C^n \arrow[r, "f^n"] \arrow[d, two heads, "\pi_{\ker \delta_C^n}"] & I^n \arrow[d, two heads, "\pi_{\ker \delta_I^n}"] \\
C^n / \ker \delta_C^n \arrow[r, "\overline{f^n}"] & I^n / \ker \delta_I^n
\end{tikzcd}$$

Consider now the following diagram:

$$\begin{tikzcd}
C^{n+1} \arrow[r, dashed] \arrow[d, shift left, "\pi_{\ker \delta_C^n} \circ s^n"] & I^{n+1} \\
C^n / \ker \delta_C^n \arrow[u, shift left, "\overline{\delta_C^n}"] \arrow[r, "\overline{f^n}"'] & I^n / \ker \delta_I^n \arrow[u, "\overline{\delta_I^n}"']
\end{tikzcd}$$

We have

$$\begin{aligned}
(\pi_{\ker \delta_C^n} \circ s^n \circ \overline{\delta_C^n}) \circ \pi_{\ker \delta_C^n} &= \pi_{\ker \delta_C^n} \circ (s^n \circ \delta_C^n) \\
&= \pi_{\ker \delta_C^n} \circ (\mathrm{id}_{C^n} - \delta_C^{n-1} \circ s^{n-1}) \\
&= \mathrm{id}_{C^n / \ker \delta_C^n} \circ \pi_{\ker \delta_C^n}.
\end{aligned}$$

Hence, $\pi_{\ker \delta_C^n} \circ s^n \circ \overline{\delta_C^n} = \mathrm{id}_{C^n / \ker \delta_C^n}$. Also

$$\|\pi_{\ker \delta_C^n} \circ s^n\|_\infty \leq \|\pi_{\ker \delta_C^n}\|_\infty \cdot \|s^n\|_\infty \leq 1,$$

and thus $\overline{\delta_C^n}$ is relatively injective. Therefore, we can find a bounded $\mathcal{G}$-map f^{n+1} solving the corresponding lifting problem.

We see directly that this is the desired part of the cochain map since

$$\begin{aligned}
f^{n+1} \circ \delta_C^n &= f^{n+1} \circ \overline{\delta_C^n} \circ \pi_{\ker \delta_C^n} \\
&= \overline{\delta_I^n} \circ \overline{f^n} \circ \pi_{\ker \delta_C^n} \\
&= \overline{\delta_I^n} \circ \pi_{\ker \delta_I^n} \circ f^n \\
&= \delta_I^n \circ f^n.
\end{aligned}$$

The uniqueness can be seen by constructing a $\mathcal{G}$-cochain homotopy using similar arguments. □

Corollary 3.4.8. Let $\mathcal{G}$ be a groupoid and V a $\mathcal{G}$-module. Then there exists an up to canonical bounded $\mathcal{G}$-cochain homotopy equivalence unique strong relatively injective $\mathcal{G}$-resolution of V.

Correspondingly, with a proof completely dual to the one for Theorem 3.4.7:

Proposition 3.4.9 (Fundamental lemma II)**.** *Let $\mathcal{G}$ be a groupoid. Let (P_*, ∂_*^P) be a relatively projective Banach $\mathcal{G}$-cochain complex and $\varepsilon\colon P_0 \longrightarrow V$ a $\mathcal{G}$-augmentation map. Let $(C_*, \partial_*^C, \nu\colon C_0 \longrightarrow W)$ be a strong $\mathcal{G}$-resolution of a Banach $\mathcal{G}$-module W.*

Let $f\colon V \longrightarrow W$ be a $\mathcal{G}$-morphism. Then there exists an up to bounded $\mathcal{G}$-chain homotopy unique extension of f to a bounded $\mathcal{G}$-chain map between the augmented chain complex $(P_*, \partial_*^P, \varepsilon)$ and the resolution (C_*, ∂_*^C, ν).

Theorem 3.4.10 (Bounded groupoid cohomology via relative homological algebra)**.** *Let $\mathcal{G}$ be a groupoid and V a Banach $\mathcal{G}$-module. Let $((D^*, \delta_D^*), \varepsilon\colon V \longrightarrow D^0)$ be a strong $\mathcal{G}$-resolution of V.*

Then for each strong cochain contraction of (D^*, ε) there exists a canonical norm non-increasing cochain map of this resolution to the standard resolution $(B(C_n(\mathcal{G}), V))_{n\in\mathbb{N}}$ of V extending id_V.

Proof. The argument is similar to the proof of Proposition 3.4.3, see also [48, Lemma 3.2.2] for the group case.

Let $((s^n\colon D^n \longrightarrow D^{n-1})_{n\in\mathbb{N}}, s^0\colon D^0 \longrightarrow V)$ be a norm non-increasing cochain contraction of the augmented cochain complex. We will define families $(\alpha_e^n\colon D_e^n \longrightarrow B(C_n(\mathcal{G})_e, V_e))_{e\in \mathrm{ob}\,\mathcal{G}}$ by induction over $n \in \mathbb{N} \cup \{-1\}$. First, we set $\alpha^{-1} = \mathrm{id}_V$. Assume we have defined α_{n-1} for some $n \in \mathbb{N}$. Then we set for all $(g_0, \dots, g_n) \in P_n(\mathcal{G})$ and all $\varphi \in D_{t(g_0)}^n$

$$\alpha_{t(g_0)}^n(\varphi)(g_0, \dots, g_n) = \alpha_{t(g_0)}^{n-1}(g_0 \cdot s_{s(g_0)}^n(g_0^{-1} \cdot \varphi))(g_0 \cdot g_1, \dots, g_n).$$

We immediately see that this is a $\mathcal{G}$-map for all $n \in \mathbb{N}$ and since s^n is norm non-increasing and the $\mathcal{G}$-action is isometric, α^n is norm non-increasing by induction. By a short calculation we see that $(\alpha^n)_{n\in\mathbb{N}}$ is a cochain map. □

Corollary 3.4.11. Let $\mathcal{G}$ be a groupoid and V a Banach $\mathcal{G}$-module. Furthermore, let $((D^n, \delta_D^n)_{n\in\mathbb{N}}, \varepsilon\colon V \longrightarrow D^0)$ be a strong relatively injective $\mathcal{G}$-resolution of V. Then there exists a canonical semi-norm non-increasing isomorphism of graded $\mathbb{R}$-modules

$$H^*(D^{*\mathcal{G}}) \longrightarrow H_b^*(\mathcal{G}; V).$$

Proof. By Theorem 3.4.10 there exists a norm non-increasing $\mathcal{G}$-cochain map

$$D^n \longrightarrow B(C_n(\mathcal{G}), V).$$

extending id_V. This induces a norm non-increasing morphism

$$H^*(D^{*\mathcal{G}}) \longrightarrow H_b^*(\mathcal{G}; V).$$

By the fundamental lemma for groupoids, this is an isomorphism. □

Dually, we get the following result for ℓ^1-homology:

Theorem 3.4.12 (ℓ^1-homology via relative homological algebra)**.** *Let $\mathcal{G}$ be a groupoid and V a $\mathcal{G}$-module. Let $((E_*, \partial_*^E), \varepsilon\colon E_0 \longrightarrow V)$ be a strong $\mathcal{G}$-resolution of V.*

Then for each strong chain contraction of (E_*, ε) there exists a canonical norm non-increasing chain map from the standard resolution $(C_*(\mathcal{G})\overline{\otimes}V)$ of V to (E_*, ε) extending id_V.

Proof. The proof is similar to the proof of Theorem 3.4.10:

Let $((s_n : E_n \longrightarrow E_{n+1})_{n\in\mathbb{N}}, s_{-1} : V \longrightarrow E_0)$ be a norm non-increasing chain contraction of the augmented chain complex. By induction over $n \in \mathbb{N} \cup \{-1\}$, we define a bounded $\mathcal{G}$-chain map $(\alpha_n : C_n(\mathcal{G})\overline{\otimes}V \longrightarrow E_n)_{n\in\mathbb{N}}$. First, we set $\alpha_{-1} = \mathrm{id}_V$. Assume we have defined α_{n-1} for some $n \in \mathbb{N} \cup \{-1\}$. Then we set for all $(g_0, \ldots, g_n) \in P_n(\mathcal{G})$ and all $v \in V_{t(g_0)}$

$$\alpha_n((g_0, \ldots, g_n) \otimes v) = g_0 \cdot s_{n-1}(\alpha_{n-1}((g_1, \ldots, g_n) \otimes g_0^{-1} \cdot v)).$$

We immediately see that this is a $\mathcal{G}$-map for all $n \in \mathbb{N}$ and since s_{n-1} is norm non-increasing and the $\mathcal{G}$-action is isometric, α_n is norm non-increasing by induction. By a short calculation we see that $(\alpha_n)_{n\in\mathbb{N}}$ is a chain map extending id_V. □

Corollary 3.4.13. Let $\mathcal{G}$ be a groupoid and V a Banach $\mathcal{G}$-module. Furthermore, let $((E_*, \partial_*^E), \varepsilon : E_0 \longrightarrow V)$ be a strong relatively projective $\mathcal{G}$-resolution of V. Then there exists a canonical semi-norm non-increasing isomorphism of graded $\mathbb{R}$-modules

$$H_*^{\ell^1}(\mathcal{G}; V) \longrightarrow H_*(E_{*\mathcal{G}}).$$

Proof. As for Corollary 3.4.11. □

3.5 Bounded Cohomology for Pairs of Groupoids

3.5.1 Relative Bounded Cohomology

We will now define relative bounded cohomology for pairs of groupoids. Using this definition, we derive a long exact sequence for bounded cohomology of groupoids. Then, we show that relative bounded cohomology is a homotopy invariant. Finally, we discuss the special case of a group relative to a family of subgroups and get a long exact sequence relating bounded cohomology of a group relative to a finite family of subgroups to the regular bounded homology of the group and the subgroups. As before, we will also discuss relative ℓ^1-homology where analogous results hold.

Though we do not elaborate relative (co-)homology for groupoid pairs, the definitions and results here can easily be adapted to the non-normed situation.

Definition 3.5.1. Let $i : \mathcal{A} \hookrightarrow \mathcal{G}$ be a groupoid pair, i.e. the canonical inclusion of a subgroupoid $\mathcal{A} \subset \mathcal{G}$. Let V be a $\mathcal{G}$-module.

(i) The map

$$\begin{aligned} C_*(i; V) : C_*(\mathcal{A}; i^*V) &\longrightarrow C_*(\mathcal{G}; V) \\ a \otimes v &\longmapsto a \otimes v \end{aligned}$$

is injective and induces an injection $C_*^{\ell^1}(\mathcal{A}; i^*V) \longrightarrow C_*^{\ell^1}(\mathcal{G}; V)$. The quotient induced by this map

$$C_*^{\ell^1}(\mathcal{G}, \mathcal{A}; V) := C_*^{\ell^1}(\mathcal{G}; V)/C_*^{\ell^1}(\mathcal{A}; i^*V),$$

endowed with the induced norm on the quotient, is a normed $\mathbb{R}$-chain complex. We write $p_*^{(\mathcal{G},\mathcal{A};V)} : C_*^{\ell^1}(\mathcal{G}; V) \longrightarrow C_*^{\ell^1}(\mathcal{G}, \mathcal{A}; V)$ for the canonical projection.

(ii) We call
$$H_*^{\ell^1}(\mathcal{G},\mathcal{A};V) := H_*(C_*^{\ell^1}(\mathcal{G},\mathcal{A};V))$$
the *ℓ^1-homology of $\mathcal{G}$ relative $\mathcal{A}$ with coefficients in V.*

(iii) Dually, the map
$$\begin{aligned} C_b^*(i;V)\colon C_b^*(\mathcal{G};V) &\longrightarrow C_b^*(\mathcal{A},i^*V)\\ (f_e)_{e\in \operatorname{ob}\mathcal{G}} &\longmapsto (f_e|_{C_*(\mathcal{A})_e})_{e\in\operatorname{ob}\mathcal{A}} \end{aligned}$$
is surjective. Its kernel
$$\begin{aligned} C_b^*(\mathcal{G},\mathcal{A};V) :=& \ker(C_b^*(i;V))\\ =&\{f\in B_{\mathcal{G}}(C_*(\mathcal{G}),V) \mid \forall_{e\in\operatorname{ob}\mathcal{A}} \quad f_e|_{C_*(\mathcal{A})_e}=0\}, \end{aligned}$$
endowed with the induced norm on the subspace, is a normed $\mathbb{R}$-cochain complex. We write $\iota^*_{(\mathcal{G},\mathcal{A};V)}\colon C_b^*(\mathcal{G},\mathcal{A};V)\longrightarrow C_b^*(\mathcal{G};V)$ for the canoncial inclusion.

(iv) We call
$$H_b^*(\mathcal{G},\mathcal{A};V) := H^*(C_b^*(\mathcal{G},\mathcal{A};V))$$
the *bounded cohomology of $\mathcal{G}$ relative to $\mathcal{A}$ with coefficients in V.*

We define domain categories $\mathsf{Grp^2Ban}$ and $\mathsf{Grp^2\overline{Ban}}$ analogously to the definition of GrpBan and $\mathsf{Grp\overline{Ban}}$ by considering pairs of groupoids instead of groupoids.

Definition 3.5.2.

(i) Let $(f,\varphi)\colon ((\mathcal{G},\mathcal{A}),V)\longrightarrow((\mathcal{H},\mathcal{B}),W)$ be a morphism in $\mathsf{Grp^2Ban}$. Then $C_*(f;\varphi)$ induces a bounded chain map
$$C_*^{\ell^1}(f,f|_{\mathcal{A}};\varphi)\colon C_*^{\ell^1}(\mathcal{G},\mathcal{A};V)\longrightarrow C_*^{\ell^1}(\mathcal{H},\mathcal{B};W);$$
and a continuous map between graded normed modules
$$H_*^{\ell^1}(f,f|_{\mathcal{A}};\varphi)\colon H_*^{\ell^1}(\mathcal{G},\mathcal{A};V)\longrightarrow H_*^{\ell^1}(\mathcal{H},\mathcal{B};W).$$
This defines a functor $\mathsf{Grp^2Ban}\longrightarrow \mathbb{R}\text{-}\mathsf{Mod}_*^{\|\cdot\|}$.

(ii) Let $(f,\varphi)\colon ((\mathcal{G},\mathcal{A}),V)\longrightarrow((\mathcal{H},\mathcal{B}),W)$ be a morphism in $\mathsf{Grp^2\overline{Ban}}$. Then $C_b^*(f,\varphi)$ restricts to a bounded cochain map
$$C_b^*(f,f|_{\mathcal{A}};\varphi) := C_b^*(f;\varphi)|_{C_b^*(\mathcal{H},\mathcal{B};W)}\colon C_b^*(\mathcal{H},\mathcal{B};W)\longrightarrow C_b^*(\mathcal{G},\mathcal{A};V).$$
This induces a continuous map between graded normed modules
$$H_b^*(f,f|_{\mathcal{A}};\varphi)\colon H_b^*(\mathcal{H},\mathcal{B};W)\longrightarrow H_b^*(\mathcal{G},\mathcal{A};V).$$
This defines a contravariant functor $\mathsf{Grp^2\overline{Ban}}\longrightarrow \mathbb{R}\text{-}\mathsf{Mod}_*^{\|\cdot\|}$.

As for the above definition, we just note that the right-hand square in the following diagram commutes by definition, so the map on the left-hand side is defined:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C_b^*(\mathcal{H},\mathcal{B};W) & \xrightarrow{\iota^*_{(\mathcal{H},\mathcal{B};W)}} & C_b^*(\mathcal{H};W) & \xrightarrow{C_b^*(i_\mathcal{B};V)} & C_b^*(\mathcal{B};i_\mathcal{B}^*W) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle C_b^*(f,f|_\mathcal{A};\varphi)} & & \downarrow{\scriptstyle C_b^*(f;\varphi)} & & \downarrow{\scriptstyle C_b^*(f|_\mathcal{A};i_\mathcal{A}^*\varphi)} & & \\
0 & \longrightarrow & C_b^*(\mathcal{G},\mathcal{A};V) & \xrightarrow[\iota^*_{(\mathcal{G},\mathcal{A};V)}]{} & C_b^*(\mathcal{G};V) & \xrightarrow[C_b^*(i_\mathcal{A};V)]{} & C_b^*(\mathcal{A};i_\mathcal{A}^*V) & \longrightarrow & 0
\end{array}$$

Here $i_\mathcal{A}^*\varphi\colon f|_\mathcal{A}^* i_\mathcal{B}^* W = i_\mathcal{A}^* f^* V \longrightarrow i_\mathcal{A}^* V$ denotes the $\mathcal{A}$-map given by $(\varphi_a)_{a\in \operatorname{ob}\mathcal{G}}$, i.e., by restricting the family corresponding to φ to objects in $\mathcal{A}$. Functoriality then follows from the functoriality of the right-hand square. Similarly for ℓ^1-homology.

In particular, we can apply the snake lemma to get the following:

Proposition 3.5.3. *Let $i_\mathcal{A}\colon \mathcal{A} \longrightarrow \mathcal{G}$ be the inclusion of a subgroupoid .*

(i) There is a natural (with respect to morphisms in $\mathsf{Grp}^2\mathsf{Ban}$) long exact sequence

$$\cdots \longrightarrow H_*^{\ell^1}(\mathcal{A};i_\mathcal{A}^*V) \xrightarrow{H_*^{\ell^1}(i_\mathcal{A};V)} H_*^{\ell^1}(\mathcal{G};V) \xrightarrow{H_*(p_*^{(\mathcal{G},\mathcal{A};V)})} H_*^{\ell^1}(\mathcal{G},\mathcal{A};V) \xrightarrow{\partial_*} H_{*-1}^{\ell^1}(\mathcal{A};i_\mathcal{A}^*V) \longrightarrow \cdots$$

such that ∂_* is continuous with respect to the induced semi-norms.

(ii) There is a natural (with respect to morphisms in $\mathsf{Grp}^2\overline{\mathsf{Ban}}$) long exact sequence

$$\cdots \longrightarrow H_b^*(\mathcal{G},\mathcal{A};V) \xrightarrow{H^*(\iota^*_{(\mathcal{G},\mathcal{A};V)})} H_b^*(\mathcal{G};V) \xrightarrow{H_b^*(i_\mathcal{A};V)} H_b^*(\mathcal{A};i_\mathcal{A}^*V) \xrightarrow{\delta^*} H_b^{*+1}(\mathcal{G},\mathcal{A};V) \longrightarrow \cdots$$

such that δ^* is continuous with respect to the induced semi-norms.

Definition 3.5.4. Let $f,g\colon (\mathcal{G},\mathcal{A}) \longrightarrow (\mathcal{H},\mathcal{B})$ be maps of pairs of groupoids. We call a homotopy h between f and g a *homotopy relative* $\mathcal{A}$ if for all $a \in \mathcal{A}$ the morphism $H_a\colon f(a) \longrightarrow g(a)$ is contained in $\mathcal{B}$.

In this situation, we also write $f \simeq_{\mathcal{A},\mathcal{B}} g$. The restriction of h induces then a homotopy $h|_\mathcal{A}$ between $f|_\mathcal{A}, g|_\mathcal{A}\colon \mathcal{A} \longrightarrow \mathcal{B}$.

We can give an equivalent description of relative homotopies using Proposition 3.1.9:

Remark 3.5.5. Let $f,g\colon (\mathcal{G},\mathcal{A}) \longrightarrow (\mathcal{H},\mathcal{B})$ be maps of pairs of groupoids and h a homotopy between f and g. Let $H\colon \mathcal{G}\times\Delta^1 \longrightarrow \mathcal{H}$ be the induced groupoid map, as in Proposition 3.1.9. Then h is a homotopy relative $\mathcal{A}$ if and only if H restricts to a groupoid map $\mathcal{A}\times\Delta^1 \longrightarrow \mathcal{B}$.

Proposition 3.5.6.

(i) Let $f,g\colon (\mathcal{G},\mathcal{A}) \longrightarrow (\mathcal{H},\mathcal{B})$ be maps of pairs of groupoids and V be an $\mathcal{H}$-module. If $f \simeq_{\mathcal{A},\mathcal{B}} g$ via a homotopy h, then there is a canonical $\mathbb{R}$-cochain homotopy between $C_b^*(f,f|_\mathcal{A};V)$ and $C_b^*(g,g|_\mathcal{A};V\circ\overline{h})$.

(ii) In particular,

$$H_b^*(f, f|_{\mathcal{A}}; V) = H_b^*(\mathrm{id}_{\mathcal{G}}, \mathrm{id}_{\mathcal{A}}; V \circ \overline{h}) \circ H_b^*(g, g|_{\mathcal{A}}; V);$$

and the map $H_b^*(\mathrm{id}_{\mathcal{G}}, \mathrm{id}_{\mathcal{A}}; V \circ \overline{h})$ *is an isometric isomorphism.*

Proof.

(i) We write s_*^h and $s_*^{h|_{\mathcal{A}}}$ for the $\mathcal{G}$-chain homotopies constructed in Proposition 3.2.30 (i) with respect to h and $h|_{\mathcal{A}}$. Let $s^*_{V,h}$ and $s^*_{i_{\mathcal{B}}^* V, h|_{\mathcal{A}}}$ denote the cochain homotopies induced by s_*^h and $s_*^{h|_{\mathcal{A}}}$. The right-hand side square of the following diagram

$$\begin{tikzcd}
0 \arrow[r] & C_b^*(\mathcal{H}, \mathcal{B}; V) \arrow[r, "\iota^*_{(\mathcal{H},\mathcal{B};V)}"] \arrow[d, "s^*_{V,h}|_{C^*(\mathcal{H},\mathcal{B};V)}"] & C_b^*(\mathcal{H}; V) \arrow[r, "C_b^*(i_{\mathcal{B}}; V)"] \arrow[d, "s^*_{V,h}"] & C_b^*(\mathcal{B}; i_{\mathcal{B}}^* V) \arrow[r] \arrow[d, "s^*_{i_{\mathcal{B}}^* V, h|_{\mathcal{A}}}"] & 0 \\
0 \arrow[r] & C_b^{*-1}(\mathcal{G}, \mathcal{A}; f^*V) \arrow[r, "\iota^{*-1}_{(\mathcal{G},\mathcal{A};f^*V)}"'] & C_b^{*-1}(\mathcal{G}; f^*V) \arrow[r, "C_b^{*-1}(i_{\mathcal{A}}; f^*V)"'] & C_b^{*-1}(\mathcal{A}; i_{\mathcal{A}}^* f^*V) \arrow[r] & 0.
\end{tikzcd}$$

commutes by definition, since for all $(\psi_e)_{e \in \mathrm{ob}\,\mathcal{H}} \in C_b^*(\mathcal{H}; V)$

$$\begin{aligned}
(C_b^{*-1}(i_a; V) \circ s^*_{V,h})(\psi_e)_{e \in \mathcal{H}} &= (\psi_{f(e)} \circ s^h_{*,e})_{e \in \mathrm{ob}\,\mathcal{A}} \\
&= (\psi_{f|_{\mathcal{A}}(e)} \circ s^{h|_{\mathcal{A}}}_{*,e})_{e \in \mathrm{ob}\,\mathcal{A}} \\
&= s^*_{i_{\mathcal{B}}^* V, h|_{\mathcal{A}}}(\psi_e)_{e \in \mathrm{ob}\,\mathcal{B}} \\
&= (s^*_{i_{\mathcal{B}}^* V, h|_{\mathcal{A}}} \circ C_b^*(i_{\mathcal{B}}; V))(\psi_e)_{e \in \mathrm{ob}\,\mathcal{H}}.
\end{aligned}$$

Hence the map on the left-hand side is defined. The map $s^*_{i_{\mathcal{B}}^* V, h|_{\mathcal{A}}}$ is a cochain homotopy between

$$\begin{aligned}
C_b^*(f|_{\mathcal{A}}, i_{\mathcal{B}}^* V) &= C_b^*(f|_{\mathcal{A}}, \mathrm{id}_{f|_{\mathcal{A}}^* i_{\mathcal{B}}^* V}) \\
&= C_b^*(f|_{\mathcal{A}}, \mathrm{id}_{i_{\mathcal{A}}^* f^* V}) \\
&= C_b^*(f|_{\mathcal{A}}, i_{\mathcal{A}}^* \,\mathrm{id}_{f^* V})
\end{aligned}$$

and

$$C_b^*(g|_{\mathcal{A}}, (V \circ \overline{h|_{\mathcal{A}}})) = C_b^*(g|_{\mathcal{A}}, i_{\mathcal{A}}^*(V \circ \overline{h})).$$

Hence, by comparing the diagrams defining $C_b^*(f, f|_{\mathcal{A}}; V)$, $C_b^*(g, g|_{\mathcal{A}}; V)$ and $s^*_{V,h}|_{C^*(\mathcal{H},\mathcal{B};V)}$, we see that $s^*_{V,h}|_{C^*(\mathcal{H},\mathcal{B};V)}$ defines an $\mathbb{R}$-cochain homotopy between $C_b^*(f, f|_{\mathcal{A}}; V)$ and $C_b^*(g, g|_{\mathcal{A}}; V)$.

(ii) The map $C_b^*(\mathrm{id}_{\mathcal{G}}, \mathrm{id}_{\mathcal{A}}; V \circ \overline{h})$ is an isometric cochain isomorphism, since $C_b^*(\mathrm{id}_{\mathcal{G}}; V \circ \overline{h})$ and $C_b^*(\mathrm{id}_{\mathcal{A}}; i_{\mathcal{A}}^*(V \circ \overline{h})) = C_b^*(\mathrm{id}_{\mathcal{A}}; i_{\mathcal{A}}^* V \circ \overline{h|_{\mathcal{A}}})$ are isometric cochain isomorphisms. □

Corollary 3.5.7. Let $f\colon (\mathcal{G}, \mathcal{A}) \longrightarrow (\mathcal{H}, \mathcal{B})$ be an equivalence relative $\mathcal{A}$, i.e., there exists a map $g\colon (\mathcal{H}, \mathcal{B}) \longrightarrow (\mathcal{G}, \mathcal{A})$, such that $g \circ f \simeq_{\mathcal{A},\mathcal{B}} \mathrm{id}_G$ and $f \circ g \simeq_{\mathcal{B},\mathcal{A}} \mathrm{id}_{\mathcal{H}}$. Then

$$H_b^*(f, f|_{\mathcal{A}}; V)\colon H_b^*(\mathcal{H}, \mathcal{B}; V) \longrightarrow H_b^*(\mathcal{G}, \mathcal{A}; f^*V)$$

is an isometric isomorphism.

Remark 3.5.8. Results analogous to Proposition 3.5.6 and Corollary 3.5.7 hold by similar arguments also for ℓ^1-homology.

Proposition 3.5.9. *Let* $(\mathcal{G}, \mathcal{A})$ *be a pair of groupoids and* $\pi_0(\mathcal{G}) = \pi_0(\mathcal{A})$. *Let* $i\colon \amalg_{e\in\pi_0\mathcal{A}} (\mathcal{G}_e, \mathcal{A}_e) \longrightarrow (\mathcal{G}, \mathcal{A})$ *be the canonical inclusion. Then there exists a groupoid map* $p\colon (\mathcal{G}, \mathcal{A}) \longrightarrow \amalg_{e\in\pi_0\mathcal{A}}(\mathcal{G}_e, \mathcal{A}_e)$ *such that* $p\circ i = \mathrm{id}_{\mathcal{A}}$ *and* $i\circ p \simeq_{\mathcal{A},\mathcal{B}} \mathrm{id}_{\mathcal{G}}$.

Proof. We proceed as in the proof of Theorem 3.1.7, constructing α, e and p but under the additional condition that

$$\forall_{a\in \operatorname{ob}\mathcal{A}} \quad e(a) \in \operatorname{Hom}_{\mathcal{A}}(a, \alpha(a)).$$

(We can achieve this because $\pi_0(\mathcal{G}) = \pi_0(\mathcal{A})$.) Then, by definition p maps $\mathcal{A}$ to $\amalg_{e\in\pi_0\mathcal{A}}\mathcal{A}$ and e is a homotopy relative $\mathcal{A}$ between $i \circ p$ and $\mathrm{id}_{\mathcal{G}}$. ☐

Corollary 3.5.10. Let $(\mathcal{G}, \mathcal{A})$ be a pair of connected groupoids with vertex group G and A respectively. Then we get an isometric isomorphism

$$H_b^*(\mathcal{G}, \mathcal{A}; \mathbb{R}[\mathcal{G}]) \cong H_b^*(G, A; \mathbb{R}).$$

Definition 3.5.11. Let G be a group and I a set.

(i) Recall from Definition 3.1.10 that we can define a groupoid G_I by setting

- Objects: $\operatorname{ob} G_I := I$.
- Morphisms: $\forall_{e,f\in I} \quad \operatorname{Mor}_{G_I}(e, f) := G$.

We then define composition by multiplication of elements in G.

(ii) If V is a G-module, we can define a G_I-module structure on

$$V_I = (V_{I,e})_{e\in I} := (V)_{e\in I}$$

by setting for all $g \in G_I$

$$\begin{aligned} \rho_g\colon V_{I,s(g)} &\longrightarrow V_{I,t(g)} \\ v &\longmapsto g \cdot v. \end{aligned}$$

Definition 3.5.12. Let G be a group and $(A_i)_{i\in I}$ be a family of subgroups. Let V be a Banach G-module. We define

$$H_*^{\ell^1}(G, (A_i)_{i\in I}; V) := H_*^{\ell^1}(G_I, \amalg_{i\in I} A_i; V_I)$$

and

$$H_b^*(G, (A_i)_{i\in I}; V) := H_b^*(G_I, \amalg_{i\in I} A_i; V_I)$$

This is functorial in the obvious way. We will also sometimes slightly abuse notation and write $(G, (A_i)_{i\in I}; V)$ to denote $(G_I, \amalg_{i\in I} A_i; V_I)$.

Lemma 3.5.13. Let G be a group and I be a set. For each $i \in I$ let $l_i\colon G \longrightarrow G_I$ denote the canonical inclusion as a vertex group. Then

$$H_b^*(l_i; V_I) = H_b^*(l_j; V_I)$$

for all $i, j \in I$ and for all Banach G-modules V.

Proof. We can define a homotopy between l_i and l_j by setting

$$h_1 := \mathrm{id}_G \colon G = G_i \longrightarrow G_j = G.$$

Here 1 denotes the unique vertex of G. Hence, by Proposition 3.3.14

$$H_b^*(l_i; V_I) = H_b^*(\mathrm{id}_G; V_I \circ h) \circ H_b^*(l_j; V_I) = H_b^*(l_j; V_I). \qquad \square$$

Theorem 3.5.14. *Let G be a group and $(A_i)_{i \in I}$ a finite family of subgroups. Let V be a Banach G-module. Then there is a natural long exact sequence*

$$\cdots \to H_b^*(G, (A_i)_{i\in I}; V) \xrightarrow{\iota^*} H_b^*(G; V) \xrightarrow{(H_b^*(i_{A_i}; V))_{i\in I}} \prod_{i\in I} H_b^*(A_i; i_{A_i}^* V) \xrightarrow{d^*} H_b^{*+1}(G, (A_i)_{i\in I}; V) \to \cdots$$

such that δ^* is continuous with respect to the induced semi-norm and the product semi-norm. Here

$$\iota^* := H_b^*(l_e; V_I) \circ H^*(\iota_{G_I, A_I, V_I}).$$

where $l_e \colon G \longrightarrow G_I$ is the canonical inclusions for some vertex $e \in \mathrm{ob}\,\mathcal{G}$.

Proof. Write $t_j \colon A_j \hookrightarrow \amalg_{i\in I} A_i$ and $s \colon \amalg_{i \in I} A_i \hookrightarrow G_I$ for the canonical inclusions. By Corollary 3.1.8 and Proposition 3.3.18, the rows in the following diagram are isometric isomorphisms:

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] \\
H_b^*(G; (A_i)_{i\in I}; V) \arrow[r, "="] \arrow[d, "H^*(\iota^*_{(G,(A_i)_{i\in I};V)})"'] & H_b^*(G; (A_i)_{i\in I}; V) \arrow[d, "\iota^*"] \\
H_b^*(G_I; V_I) \arrow[r, "H_b^*(l_e; V_I)", "\cong"'] \arrow[d, "H_b^*(s; V_I)"'] & H_b^*(G; V) \arrow[d, "(H_b^*(i_{A_i}; V))_{i\in I}"] \\
H_b^*(\amalg_{i\in I} A_i; s^*V) \arrow[r, "(H_b^*(t_i; s^*V))_{i\in I}", "\cong"'] \arrow[d, "\delta^*"'] & \prod_{i\in I} H_b^*(A_i; i_{A_i}^* V) \arrow[d, "d^*"] \\
H_b^{*+1}(G, (A_i)_{i\in I}; V) \arrow[r, "="] \arrow[d] & H_b^{*+1}(G, (A_i)_{i\in I}; V) \arrow[d] \\
\vdots & \vdots
\end{tikzcd}$$

The upper square commutes by definition and we can choose a continuous map d^* in such a way, that the lower square commutes. The centre square commutes by Lemma 3.5.13. $\square$

3.5.2 Relative Homological Algebra for Pairs

In this section, we will discuss a version of relative homological algebra that can be used to describe the bounded cohomology of a pair of groupoids. The definitions and results will be analogous to the ones in the absolute setting in Section 3.4. If $(\mathcal{G}, \mathcal{A})$ is a pair of groupoids, we will define $(\mathcal{G}, \mathcal{A})$-cochain complexes and strong, relatively injective $(\mathcal{G}, \mathcal{A})$-resolutions in this setting. We will see that there is a fundamental lemma for pairs and that the pair of standard resolutions is a strong, relatively injective $(\mathcal{G}, \mathcal{A})$-resolution, thus $H_b^*(\mathcal{G}, \mathcal{A}; V)$ can be calculated by such resolutions. Our definition is slightly more restrictive than Park's definition of allowable pairs [68], but will directly lead to the fundamental lemma for pairs.

Definition 3.5.15. Let $i \colon \mathcal{A} \longrightarrow \mathcal{G}$ be a pair of groupoids, i.e., $\mathcal{A}$ and $\mathcal{G}$ are groupoids and i is an injective groupoid map, see Definition 3.1.1. We define a category $(\mathcal{G}, \mathcal{A})$-$\mathsf{Ban}$ by setting:

(i) Objects in $(\mathcal{G}, \mathcal{A})$-$\mathsf{Ban}$ are tripels (V, V', φ), where V is a Banach $\mathcal{G}$-module, V' a Banach $\mathcal{A}$-module and $\varphi \colon i^*V \longrightarrow V'$ an $\mathcal{A}$-morphism. We call such an object a $(\mathcal{G}, \mathcal{A})$-*module*.

(ii) A morphism $(j, j') \colon (V, V', \varphi) \longrightarrow (W, W', \psi)$ in $(\mathcal{G}, \mathcal{A})$-$\mathsf{Ban}$ is a pair (j, j'), consisting of a $\mathcal{G}$-map $j \colon V \longrightarrow W$ and an $\mathcal{A}$-map $j' \colon V' \longrightarrow W'$, such that the following diagram commutes:

$$\begin{tikzcd} i^*V^* \arrow[r, "i^*j"] \arrow[d, "\varphi"] & i^*W \arrow[d, "\psi"] \\ V' \arrow[r, "j'"] & W' \end{tikzcd}$$

Composition is then defined componentwise. We call such a morphism also a $(\mathcal{G}, \mathcal{A})$-*map*. In addition, we will consider not necessarily $(\mathcal{G}, \mathcal{A})$-equivariant morphisms $(j, j') \colon (V, V', \varphi) \longrightarrow (W, W', \psi)$ by dropping the condition that j and j' are equivariant, but still demanding that the above diagram commutes.

(iii) Similarly, we also define a category ${}_{(\mathcal{G},\mathcal{A})}\mathsf{Ch}^{\|\cdot\|}$ of Banach $(\mathcal{G}, \mathcal{A})$-cochain complexes. The notions of augmentations, cochain homotopies etc. translate naturally into this setting.

Definition 3.5.16 (Cohomology of $(\mathcal{G}, \mathcal{A})$-cochain complexes)**.** Let $i \colon \mathcal{A} \longrightarrow \mathcal{G}$ be a pair of groupoids.

(i) Let (C^*, D^*, f^*) be a $(\mathcal{G}, \mathcal{A})$-cochain complex. The map f^* restricts to a cochain map

$$\begin{aligned} \overline{f^*} \colon C^{*\mathcal{G}} &\longrightarrow D^{*\mathcal{A}} \\ (v_e)_{e \in \operatorname{ob} \mathcal{G}} &\longmapsto (f_a^*(v_{i(a)}))_{a \in \operatorname{ob} \mathcal{A}}. \end{aligned}$$

We write $K^*(C^*, D^*, f^*)$ for the normed $\mathbb{R}$-cochain complex $\ker(\overline{f^*})$, given by considering the kernel in each degree and endowed with the norm induced by the norm on C^*.

(ii) Let $(j,j')\colon (C_0^*, D_0^*, f_0^*) \longrightarrow (C_1^*, D_1^*, f_1^*)$ be a $(\mathcal{G},\mathcal{A})$-cochain map. Then, by restriction, (j,j') induce the maps on the right-hand side of the following diagram

$$\begin{tikzcd}
\ker \overline{f_0^*} \arrow[r] \arrow[d, "\overline{j^*}|_{\ker \overline{f_0^*}}"] & C_0^{*\mathcal{G}} \arrow[r, "\overline{f_0^*}"] \arrow[d, "\overline{j^*}"] & D_0^{*\mathcal{A}} \arrow[d, "\overline{j'^*}"] \\
\ker \overline{f_1^*} \arrow[r] & C_1^{*\mathcal{G}} \arrow[r, "\overline{f_1^*}"] & D_1^{*\mathcal{A}},
\end{tikzcd}$$

and this square commutes, hence the map on left-hand side is defined. We write $K^*(j,j') := \overline{j^*}|_{\ker \overline{f_0^*}}$ for the map on the left-hand side. In this way, K^* defines a functor ${}_{(\mathcal{G},\mathcal{A})}\mathsf{Ch}^{\|\cdot\|} \longrightarrow {}_{\mathbb{R}}\mathsf{Ch}^{\|\cdot\|}$.

(iii) We write $H^*(C^*, D^*, f^*)$ to denote the cohomology of $K^*(C^*, D^*, f^*)$ endowed with the induced semi-norm. In this way, we have defined a functor ${}_{(\mathcal{G},\mathcal{A})}\mathsf{Ch}^{\|\cdot\|} \longrightarrow \mathbb{R}\text{-}\mathsf{Mod}_*^{\|\cdot\|}$.

The main example of $(\mathcal{G},\mathcal{A})$-cochain complexes in this section is given by the pair of canonical resolutions for $\mathcal{G}$ and $\mathcal{A}$:

Example 3.5.17. Let $i\colon \mathcal{A} \longrightarrow \mathcal{G}$ be a groupoid pair and V a Banach $\mathcal{G}$-module. Then $C^*(\mathcal{G},\mathcal{A};V) := (B(C_*(\mathcal{G}),V), B(C_*(\mathcal{A}), i^*V), B(C_*(i),V))$ is a Banach $(\mathcal{G},\mathcal{A})$-cochain complex. By definition

$$H^*(C^*(\mathcal{G},\mathcal{A};V)) = H_b^*(\mathcal{G},\mathcal{A};V).$$

Remark 3.5.18. Let $(f_0^*, f_1^*), (g_0^*, g_1^*)\colon (C_0^*, C_1^*, \varphi^*) \longrightarrow (D_0^*, D_1^*, \psi^*)$ be a pair of $(\mathcal{G},\mathcal{A})$-cochain maps. Let $(h_0^*, h_1^*)\colon (C_0^*, C_1^*, \varphi_0^*) \longrightarrow (D_0^{*-1}, D_1^{*-1}, \psi^{*-1})$ be a $(\mathcal{G},\mathcal{A})$-cochain homotopy between (f_0^*, f_1^*) and (g_0^*, g_1^*), i.e., h_0^* is a $\mathcal{G}$-cochain homotopy between f_0^* and g_0^* and h_1^* is an $\mathcal{A}$-cochain homotopy between f_1^* and g_1^*, and the pair (h_0^*, h_1^*) is a family of $(\mathcal{G},\mathcal{A})$-maps. Then (h_0^*, h_1^*) induces an $\mathbb{R}$-cochain homotopy between $K^*(f_0^*, f_1^*)$ and $K^*(g_0^*, g_1^*)$. In this sense, cohomology of $(\mathcal{G},\mathcal{A})$-cochain complexes is a homotopy invariant.

Definition 3.5.19 (Relatively injective pairs)**.** Let $i\colon \mathcal{A} \longrightarrow \mathcal{G}$ be a pair of groupoids.

(i) A $(\mathcal{G},\mathcal{A})$-map $(j,j')\colon (V,V',\varphi) \longrightarrow (W,W',\psi)$ is called *relatively injective*, if there is a (not necessarily $(\mathcal{G},\mathcal{A})$-equivariant) *split*

$$(\sigma,\sigma')\colon (W,W',\psi) \longrightarrow (V,V',\varphi),$$

such that $(\sigma,\sigma')\circ(j,j') = (\mathrm{id}_V, \mathrm{id}_{V'})$ and $\|\sigma\|_\infty \le 1$ and $\|\sigma'\|_\infty \le 1$.

(ii) A $(\mathcal{G},\mathcal{A})$-module (I,I',f) is called *relatively injective* if for each relatively injective $(\mathcal{G},\mathcal{A})$-map $(j,j')\colon (V,V',\varphi) \longrightarrow (W,W',\psi)$ between $(\mathcal{G},\mathcal{A})$-modules and each $(\mathcal{G},\mathcal{A})$-map $(\alpha,\alpha')\colon (V,V',\varphi) \longrightarrow (I,I',f)$, there exists a $(\mathcal{G},\mathcal{A})$-map $(\beta,\beta')\colon (W,W',\psi) \longrightarrow (I,I',f)$, such that $(\beta,\beta')\circ(j,j') = (\alpha,\alpha')$ and $\|\beta\|_\infty \le \|\alpha\|_\infty$ and $\|\beta'\|_\infty \le \|\alpha'\|_\infty$.

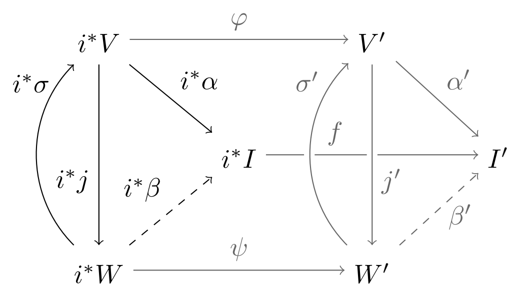

Figure 3.3: Extension Problem for Pairs.

Remark 3.5.20. Park [68] treats bounded cohomology of a pair (G, A) of groups via so called *allowable resolutions*, i.e., via pairs consisting of a strong, relatively injective G-resolution and a strong, relatively injective A-resolution, together with a cochain map that commutes with a pair of norm non-increasing cochain contractions of the two resolutions. Thus, the condition on relatively injectivity of an allowable pair is a priori weaker than our definition of relatively injective (G, A)-resolutions. However, as noted by Frigerio and Pagliantini [40, Remark 3.8], it is not clear why Park's definition should lead to a version of the fundamental lemma. Our definition avoids this problem and still includes the interesting examples.

Example 3.5.21. Let $i\colon \mathcal{A} \longrightarrow \mathcal{G}$ be a pair of groupoids and U a Banach $\mathcal{G}$-module. For each $n \in \mathbb{N}$, the Banach $(\mathcal{G}, \mathcal{A})$-module $C^n(\mathcal{G}, \mathcal{A}; U)$ is relatively injective.

Proof. Let $(j, j')\colon (V, V', \varphi) \longrightarrow (W, W', \psi)$ be a relatively injective $(\mathcal{G}, \mathcal{A})$-map and let $(\sigma, \sigma')\colon (W, W', \psi) \longrightarrow (V, V', \varphi)$ be a not necessarily equivariant split for (j, j') with $\|\sigma\|_\infty \le 1$ and $\|\sigma'\|_\infty \le 1$. Moreover, let $(\alpha, \alpha')\colon (V, V', \varphi) \longrightarrow C^n(\mathcal{G}, \mathcal{A}; U)$ be a $(\mathcal{G}, \mathcal{A})$-map. As in the proof of Proposition 3.4.3, we define a $\mathcal{G}$-map $\beta\colon W \longrightarrow B(C_n(\mathcal{G}), U)$ and an $\mathcal{A}$-map $\beta'\colon W' \longrightarrow B(C_n(\mathcal{A}), i^*U)$ by setting

$$\begin{aligned} \beta\colon W &\longrightarrow B(C_n(\mathcal{G}), U) \\ w &\longmapsto \Big((g_0, \dots, g_n) \longmapsto \alpha(g_0 \cdot \sigma(g_0^{-1} \cdot w))(g_0, \dots, g_n)\Big) \end{aligned}$$

and

$$\begin{aligned} \beta'\colon W &\longrightarrow B(C_n(\mathcal{A}), i^*U) \\ w &\longmapsto \Big((a_0, \dots, a_n) \longmapsto \alpha'(a_0 \cdot \sigma'(a_0^{-1} \cdot w))(a_0, \dots, a_n)\Big). \end{aligned}$$

In particular, $(\beta, \beta') \circ (j, j') = (\alpha, \alpha')$ and $\|\beta\|_\infty \le \|\alpha\|_\infty$ and $\|\beta'\|_\infty \le \|\alpha'\|_\infty$.

We have for all $w \in i^*W$ and $(a_0, \dots, a_0) \in C_n(\mathcal{A})$

$$\begin{aligned}
\beta'(\psi(w))(a_0, \dots, a_n) &= \alpha'(a_0 \cdot \sigma'(a_0^{-1} \cdot \psi(w)))(a_0, \dots, a_n) \\
&= \alpha'(a_0 \cdot \varphi i^* \sigma(a_0^{-1} \cdot w))(a_0, \dots, a_n) \\
&= B(C_n(i), U) i^* \alpha(a_0 \cdot i^* \sigma(a_0^{-1} \cdot w))(a_0, \dots, a_n) \\
&= B(C_n(i), U) i^* \beta(w)(a_0, \dots, a_n).
\end{aligned}$$

Hence, (β, β') is a $(\mathcal{G}, \mathcal{A})$-map. ☐

Definition 3.5.22. Let $i \colon \mathcal{A} \longrightarrow \mathcal{G}$ be a pair of groupoids, (C^*, D^*, f^*) be a Banach $(\mathcal{G}, \mathcal{A})$-cochain complex and V be a Banach $\mathcal{G}$-module. Furthermore, let $(\varepsilon, \varepsilon') \colon (V, i^*V, \mathrm{id}_{i^*V}) \longrightarrow (C^0, D^0, f^0)$ be a $(\mathcal{G}, \mathcal{A})$-augmentation map. We call $(C^*, D^*, f^*, (\varepsilon, \varepsilon'))$ a *strong resolution of* V if there exists a (not necessarily $(\mathcal{G}, \mathcal{A})$-equivariant) norm non-increasing cochain contraction (s^*, t^*) of $(C^*, D^*, f^*, (\varepsilon, \varepsilon'))$, i.e., a norm non-increasing contraction s^* of (C^*, ε) and a norm non-increasing contraction t^* of (D^*, ε'), such that f^* commutes with i^*s^* and t^*.

Let $i \colon \mathcal{A} \longrightarrow \mathcal{G}$ be a groupoid pair and V a $\mathcal{G}$-module. Recall that there is a $\mathcal{G}$-augmentation map

$$\begin{aligned}
\varepsilon \colon V &\longrightarrow B(C_0(\mathcal{G}), V) \\
v &\longmapsto (g \longmapsto v)
\end{aligned}$$

and an $\mathcal{A}$-augmentation map

$$\begin{aligned}
\varepsilon' \colon i^*V &\longrightarrow B(C_0(\mathcal{A}), i^*V) \\
v &\longmapsto (a \longmapsto v).
\end{aligned}$$

This induces a canonical $(\mathcal{G}, \mathcal{A})$-augmentation map

$$(\varepsilon, \varepsilon') \colon (V, i^*V, \mathrm{id}_{i^*V}) \longrightarrow C^0(\mathcal{G}, \mathcal{A}; V).$$

Example 3.5.23. Let $i \colon \mathcal{A} \longrightarrow \mathcal{G}$ be a pair of groupoids and V be a Banach $\mathcal{G}$-module. Let $s^*_{\mathcal{G}}$ and $s^*_{\mathcal{A}}$ be the norm-non-increasing cochain contractions for $(B(C_*(\mathcal{G}), V), \varepsilon)$ and $(B(C_*(\mathcal{A}), i^*V), \varepsilon')$ as in Example 3.4.6. By a short calculation, we see that $(s^*_{\mathcal{G}}, s^*_{\mathcal{A}})$ is a norm non-increasing cochain contraction of $(C^*(\mathcal{G}, \mathcal{A}; V), (\varepsilon, \varepsilon'))$. Hence, $(C^*(\mathcal{G}, \mathcal{A}; V), (\varepsilon, \varepsilon'))$ is a strong $(\mathcal{G}, \mathcal{A})$-resolution of V, called *the standard* $(\mathcal{G}, \mathcal{A})$*-resolution of* V.

Proposition 3.5.24 (Fundamental Lemma for Pairs)**.** *Let* $i \colon \mathcal{A} \longrightarrow \mathcal{G}$ *be a pair of groupoids. Let* (I^*, J^*, φ^*) *be a relatively injective* $(\mathcal{G}, \mathcal{A})$*-cochain complex and* $(\varepsilon, \varepsilon') \colon (W, i^*W, \mathrm{id}_{i^*W}) \longrightarrow (I^0, J^0, \varphi^0)$ *a* $(\mathcal{G}, \mathcal{A})$*-augmentation map. Let* $(C^*, D^*, \psi^*, (\nu, \nu'))$ *be a strong* $(\mathcal{G}, \mathcal{A})$*-resolution of a Banach* $\mathcal{G}$*-module* V*. Let* $(f, f') \colon (V, i^*V, \mathrm{id}_{i^*V}) \longrightarrow (W, i^*W, \mathrm{id}_{i^*W})$ *be a* $(\mathcal{G}, \mathcal{A})$*-map. Then there exists an extension of* (f, f') *to a bounded* $(\mathcal{G}, \mathcal{A})$*-cochain map from the resolution* $(C^*, D^*, \psi^*, (\nu, \nu'))$ *to the augmented cochain complex* $(I^*, J^*, \varphi^*, (\varepsilon, \varepsilon'))$*. This extension is unique up to bounded* $(\mathcal{G}, \mathcal{A})$*-cochain homotopy.*

Proof. The proof is the same as the one of Proposition 3.4.7, just solving the extension problems for pairs instead of for single modules. ☐

Proposition 3.5.25. *Let $i\colon \mathcal{A} \longrightarrow \mathcal{G}$ be a groupoid pair and V a Banach $\mathcal{G}$-module. Let $(C^*, D^*, \varphi^*, (\nu, \nu'))$ be a strong $(\mathcal{G}, \mathcal{A})$-resolution of V. Then for each norm non-increasing cochain contraction of $(C^*, D^*, \varphi^*, (\nu, \nu'))$ there exists a canonical, norm non-increasing $(\mathcal{G}, \mathcal{A})$-cochain map*

$$(C^*, D^*, \varphi^*) \longrightarrow C^*(\mathcal{G}, \mathcal{A}; V),$$

extending $(\mathrm{id}_V, \mathrm{id}_{i^*V})$.

Proof. The proof is basically the same as the one for Theorem 3.4.10, one just has to check that the constructed pair of maps is a $(\mathcal{G}, \mathcal{A})$-map. □

Corollary 3.5.26. Let $i\colon \mathcal{A} \longrightarrow \mathcal{G}$ be a groupoid pair and V a Banach $\mathcal{G}$-module. Let $(C^*, D^*, \varphi^*, (\nu, \nu'))$ be a strong, relatively injective $(\mathcal{G}, \mathcal{A})$-resolution of V. Then there exists a canonical, semi-norm non-increasing isomorphism of graded $\mathbb{R}$-modules

$$H^*(C^*, D^*, \varphi^*) \longrightarrow H_b^*(\mathcal{G}, \mathcal{A}; V).$$

Proof. By Proposition 3.5.25, there is a norm non-increasing $(\mathcal{G}, \mathcal{A})$-cochain map

$$(C^*, D^*, \varphi^*) \longrightarrow C^*(\mathcal{G}, \mathcal{A}; V),$$

extending $(\mathrm{id}_V, \mathrm{id}_{i^*V})$. Therefore, this map induces a semi-norm non-increasing map $H^*(C^*, D^*, \varphi^*) \longrightarrow H_b^*(\mathcal{G}, \mathcal{A}; V)$. By the fundamental lemma for pairs, this map is an isomorphism in each degree and does not depend of the choice of the lift of $(\mathrm{id}_V, \mathrm{id}_{i^*V})$. □

Chapter 4

Amenable Groupoids and Transfer

Amenable groups are a major class of groups considered in geometric group theory and beyond. On the one hand, they include many interesting examples, e.g., all finite groups, all solvable groups and all groups of subexponential growth and are closed under taking extensions, subgroups, etc., [26]. On the other hand, their defining property, the existence of invariant means, makes them in many instances rather accessible.

The profound importance of amenable groups for the study of bounded cohomology comes from the fact that they are "invisible" to bounded cohomology, i.e., the bounded cohomology of amenable groups vanishes [65], see Section 4.2. This vanishing can in fact be used to completely describe amenable groups via bounded cohomology. That higher homotopy groups are Abelian and thus amenable will also be central in the proof of the (topological) mapping theorem, which can be expressed roughly as saying that the higher homotopy groups are not seen by bounded cohomology, see Chapter 5.

Amenable groups will also be important in Chapter 6, where we will study their relation to uniformly finite homology.

The concept of amenability has been fruitfully extended to measured and topological actions of groups. The action groupoid for such actions becomes a measured or topological groupoid, leading to a further extension of amenability to such groupoids, [1, 2] for an overview of these concepts. However, since we are only considering discrete groupoids, we will give a definition of amenable groupoids without additional structure that is sufficient for the latter applications. Though it is beyond the scope of this thesis, it would be very interesting to modify the constructions of the previous sections to measured and topological groupoids in order to describe amenability in these settings via bounded cohomology.

This chapter is structured as follows. In the first section,we will present our definition of amenable groupoids and give some examples. In the second section, we will show that bounded cohomology of amenable groupoids vanishes and that amenability can indeed be characterised by this vanishing. We will also prove the algebraic mapping theorem, stating that, in degree greater or equal 2, the bounded cohomology of a groupoid $\mathcal{G}$ relative to an amenable subgroupoid is isometrically isomorphic to the bounded cohomology of $\mathcal{G}$.

4.1 Amenability

In this section, we present our definition of amenability for groupoids, similar to the definition of amenability of measured groupoids, [1].

Definition 4.1.1. Let $\mathcal{G}$ be a groupoid and V a normed $\mathcal{G}$-module. We define a normed $\mathcal{G}$-module $\ell^\infty(\mathcal{G}, V)$ by:

(i) For all $e \in \operatorname{ob}\mathcal{G}$ we set $\ell^\infty(\mathcal{G}, V)_e := \ell^\infty(\mathcal{G}_e, V_e)$, endowed with the $\|\cdot\|_\infty$-norm. Here, we denote by $\mathcal{G}_e$ the set of all morphisms in $\mathcal{G}$ ending in e, i.e., $\mathcal{G}_e = t^{-1}(e)$.

(ii) We define the partial $\mathcal{G}$-action by setting for all $g \in \mathcal{G}$

$$\begin{aligned} \rho_g \colon \ell^\infty(\mathcal{G}, V)_{s(g)} &\longrightarrow \ell^\infty(G, V)_{t(g)} \\ \varphi &\longmapsto \big(h \longmapsto g \cdot \varphi(g^{-1} \cdot h)\big). \end{aligned}$$

Remark 4.1.2. If $\mathcal{G}$ is a groupoid and V a normed $\mathcal{G}$-module, there is a canonical $\mathcal{G}$-inclusion map $c_V \colon V \longrightarrow \ell^\infty(\mathcal{G}, V)$, corresponding to viewing elements in V as constant functions $\mathcal{G} \longrightarrow V$, given by setting for all $e \in \operatorname{ob}\mathcal{G}$

$$\begin{aligned} c_{Ve} \colon V_e &\longrightarrow \ell^\infty(\mathcal{G}, V)_e \\ a &\longmapsto (g \longmapsto a). \end{aligned}$$

Definition 4.1.3. Let $\mathcal{G}$ be a groupoid and V a normed $\mathcal{G}$-module.

(i) An *equivariant mean on $\mathcal{G}$ with coefficients in V* is a $\mathcal{G}$-morphism

$$m \colon \ell^\infty(\mathcal{G}, V) \longrightarrow V,$$

satisfying $\|m\|_\infty = 1$ and $m \circ c_V = \mathrm{id}_V$.

If $V = \mathbb{R}[\mathcal{G}]$, we call m simply a *(left)-invariant mean on $\mathcal{G}$*.

(ii) We call $\mathcal{G}$ *amenable* if there exists a (left-)invariant mean on $\mathcal{G}$.

This definition clearly extends the classical definition of amenability for groups [69].

Example 4.1.4. Let $(A_i)_{i\in I}$ be a family of groups. Then $\amalg_{i\in I} A_i$ is amenable if and only if all A_i are amenable groups.

Example 4.1.5. Let G be an amenable group acting on a set X. Then the action groupoid $G \ltimes X$ is amenable (Example 3.1.5).

Proof. The canonical projection

$$\begin{aligned} \pi \colon G \ltimes X &\longrightarrow G \\ (g, x) &\longmapsto g \end{aligned}$$

is a groupoid map. Clearly $\pi^*\mathbb{R} \cong_{G\ltimes X} \mathbb{R}[G \ltimes X]$, and a short calculation shows that $\pi^*\ell^\infty(G, \mathbb{R}) \cong_{G\ltimes X} \ell^\infty(G \ltimes X, \mathbb{R}[G \ltimes X])$. Therefore, if $m \colon \ell^\infty(G, \mathbb{R}) \longrightarrow \mathbb{R}$ is a left-invariant mean on G, the induced map $\pi^* m$ is a left-invariant mean on $G \ltimes X$. $\square$

Proposition 4.1.6. *Let $\mathcal{G}$ be an amenable groupoid and V a normed $\mathcal{G}$-module. Then there exists an equivariant mean on $\mathcal{G}$ with coefficients in V'*

Proof. Let $m_{\mathbb{R}}$ be a left-invariant mean on $\mathcal{G}$. Then the map

$$\begin{aligned} m_V \colon \ell^\infty(\mathcal{G}, V') &\longrightarrow V' \\ \varphi &\longmapsto \Big(v \longmapsto m_{\mathbb{R}}^n\big(g \longmapsto \varphi(g)(v)\big)\Big). \end{aligned}$$

is clearly an equivariant mean on $\mathcal{G}$ with coefficients in V'. □

4.2 The Algebraic Mapping Theorem

In this section, we show that the bounded cohomology of a groupoid $\mathcal{G}$ relative to an amenable groupoid is equal to the bounded cohomology of $\mathcal{G}$. We will also prove that the bounded cohomology of an amenable groupoid vanishes and that this characterises amenable groupoids. The techniques used are similar to the group case, [62, 65].

Let $i \colon \mathcal{A} \longrightarrow \mathcal{G}$ be a pair of groupoids. Recall that we write $L_*(\mathcal{G})$ to denote the homogeneous Bar resolution of $\mathcal{G}$, Definition 3.2.22. Let V be a Banach $\mathcal{G}$-module. Denote by $K_L^*(\mathcal{G}, \mathcal{A}; V)$ the kernel of the map

$$B(L_*(i), V) \colon B(L_*(\mathcal{G}), V) \longmapsto B(L_*(\mathcal{A}), i^*V).$$

and write $j^* \colon K_L^*(\mathcal{G}, \mathcal{A}; V) \hookrightarrow B(L_*(\mathcal{G}), V)$ to denote the canonical inclusion. Since by Proposition 3.2.23 there is a natural and canonical isometric isomorphism between $L_*(\mathcal{G})$ and $C_*(\mathcal{G})$, there is a canonical isometric isomorphism between $K_L^*(\mathcal{G}, \mathcal{A}; V)$ and $K^*(\mathcal{G}, \mathcal{A}; V)$. Thus we can also use the former to calculate bounded cohomology of the pair $(\mathcal{G}, \mathcal{A})$.

Proposition 4.2.1. *Let $\mathcal{G}$ be a groupoid. The following map is a norm non-increasing $\mathcal{G}$-chain map extending* $\mathrm{id}_{\mathbb{R}[\mathcal{G}]}$

$$\begin{aligned} \mathrm{Alt}_n \colon L_n(\mathcal{G}) &\longrightarrow L_n(\mathcal{G}) \\ (g_0, \dots, g_n) &\longmapsto \frac{1}{(n+1)!} \cdot \sum_{\sigma \in \Sigma_{n+1}} \mathrm{sgn}(\sigma) \cdot (g_{\sigma(0)}, \dots, g_{\sigma(n)}) \end{aligned}$$

For a Banach $\mathcal{G}$-module V, we write $\mathrm{Alt}_V^n \colon B(L_n(\mathcal{G}), V) \longrightarrow B(L_n(\mathcal{G}), V)$ for the corresponding dual map.

Proof. The only assertion that might want a proof is that Alt_n is a chain map. But for all $e \in \operatorname{ob} \mathcal{G}$ and $(g_0, \dots, g_n) \in (\mathcal{G}_e)^{n+1}$

$$\begin{aligned} &\mathrm{Alt}_{n-1} \partial_n (g_0, \dots, g_n) \\ &= \sum_{i=0}^n (-1)^i \cdot \frac{1}{n!} \cdot \sum_{\sigma \in \Sigma_{n+1}, \sigma(i) = i} \mathrm{sgn}(\sigma) \cdot (g_{\sigma(0)}, \dots, \widehat{g_{\sigma(i)}}, \dots, g_{\sigma(n)}) \\ &= \sum_{i=0}^n (-1)^i \cdot \frac{1}{(n+1)!} \cdot \sum_{\sigma \in \Sigma_{n+1}} \mathrm{sgn}(\sigma) \cdot (g_{\sigma(0)}, \dots, \widehat{g_{\sigma(i)}}, \dots, g_{\sigma(n)}) \\ &= \partial_n \mathrm{Alt}_n (g_0, \dots, g_n). \end{aligned}$$

□

Proposition 4.2.2. *Let $\mathcal{G}$ be a groupoid and $\mathcal{A} \subset \mathcal{G}$ be an amenable subgroupoid. Let V be a Banach $\mathcal{G}$-module and $m_{\mathcal{A}}$ be an equivariant mean on $\mathcal{A}$ with coefficients in V' (with the induced $\mathcal{A}$-module structure). For each $n \in \mathbb{N}$ and each $i \in \{0, \dots, n\}$, we define a map:*

$$\begin{aligned}
&\Phi_i^n \colon B(L_n(\mathcal{G}), V') \longrightarrow B(L_n(\mathcal{G}), V')\\
&\varphi \longmapsto \Bigg((g_0, \dots, g_n)\\
&\qquad \longmapsto \begin{cases} g_i \cdot m_{A_{s(g_i)}}\big(a_i \longmapsto g_i^{-1} \cdot \varphi(g_0, \dots, g_{i-1}, g_i \cdot a_i, g_{i+1}, \dots, g_n)\big) & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{if } s(g_i) \in \operatorname{ob} \mathcal{A} & \\ \varphi(g_0, \dots, g_n) \qquad\qquad\qquad\qquad\qquad \textit{else} & \end{cases} \Bigg)
\end{aligned}$$

For each $n \in \mathbb{N}$, set $A^n_{\mathcal{A}} := \Phi^n_0 \circ \dots \circ \Phi^n_n$. Then $A^*_{\mathcal{A}}$ is a norm non-increasing $\mathcal{G}$-cochain map extending $\mathrm{id}_{V'}$.

Proof.

(i) The map $A^*_{\mathcal{A}}$ extends $\mathrm{id}_{V'}$ because the mean of a constant function is equal to the value of the function.

(ii) It follows directly that for all $n \in \mathbb{N}_{>0}$

$$\forall_{i,j \in \{0,\dots,n\}} \quad \Phi^n_j \circ \delta^{n-1}_i = \begin{cases} \delta^{n-1}_i \circ \Phi^{n-1}_j & \text{if } i > j \\ \delta^{n-1}_i \circ \Phi^{n-1}_{j-1} & \text{if } i < j \\ \delta^{n-1}_i & \text{if } i = j. \end{cases}$$

Thus $A^*_{\mathcal{A}}$ is a cochain map.

(iii) The map $A^*_{\mathcal{A}}$ is norm non-increasing because means are norm non-increasing.

(iv) The map $A^*_{\mathcal{A}}$ is $\mathcal{G}$-equivariant, since: For any $n \in \mathbb{N}$ and $i \in \{0, \dots, n\}$, consider $(g_0, \dots, g_n) \in L_n(\mathcal{G})$, $g \in \mathcal{G}$, such that $t(g) = t(g_i)$, and $\varphi \in B(L_n(\mathcal{G})_{s(g)}, V'_{s(g)})$. We can assume $s(g_i) \in \mathcal{A}$ and have

$$\begin{aligned}
&\Phi^n_i(g \cdot \varphi)(g_0, \dots, g_n)\\
&\quad = g_i \cdot m_{A_{s(g_i)}}\big(a_i \longmapsto g_i^{-1} \cdot g \cdot \varphi(g^{-1} \cdot g_0, \dots, g^{-1} \cdot g_i \cdot a_i, \dots, g^{-1} \cdot g_n)\big)\\
&\quad = g \cdot (g^{-1} \cdot g_i) \cdot m_{A_{s(g_i)}}\big(a_i \longmapsto (g^{-1} \cdot g_i)^{-1}\\
&\qquad\qquad \cdot \varphi(g^{-1} \cdot g_0, \dots, g^{-1} \cdot g_i \cdot a_i, \dots, g^{-1} \cdot g_n)\big)\\
&\quad = g \cdot \Phi^n_i(\varphi)(g^{-1} \cdot g_0, \dots, g^{-1} \cdot g_n)\\
&\quad = (g \cdot \Phi^n_i(\varphi))(g_0, \dots, g_n).
\end{aligned}$$

□

Proposition 4.2.3. *Let $\mathcal{G}$ be a groupoid and $\mathcal{A} \subset \mathcal{G}$ be an amenable subgroupoid. Let V be a Banach $\mathcal{G}$-module. Then the composition*

$$\mathrm{Alt}^*_{V'} \circ A^*_{\mathcal{A}} \colon B(L_*(\mathcal{G}), V') \longrightarrow B(L_*(\mathcal{G}), V')$$

is a norm non-increasing $\mathcal{G}$-map extending $\mathrm{id}_{V'}$. For $n \in \mathbb{N}_{\geq 1}$, it factors through $K^n_L(\mathcal{G}, \mathcal{A}; V')$, i.e., the following diagram commutes:

$$\begin{tikzcd} B(L_n(\mathcal{G}),V') \arrow[rr, "\mathrm{Alt}^n_{V'}\circ A^n_{\mathcal{A}}"] \arrow[dr] & & B(L_n(\mathcal{G}),V') \\ & K^n_L(\mathcal{G},\mathcal{A};V') \arrow[ur, "j^n"'] & \end{tikzcd}$$

Proof. We have to show that $B(L_n(i),V')\circ \mathrm{Alt}^n_{V'}\circ A^n_{\mathcal{A}} = 0$ for all $n \in \mathbb{N}_{\geq 1}$. But for all $e \in \operatorname{ob}\mathcal{A}$ and $(a'_0,\dots,a'_n) \in \mathcal{A}^{n+1}_e$, $b'_i \in A_e$ and $\varphi \in B(L_n(\mathcal{G}),V')_e$

$$\begin{aligned}
&\Phi^n_i(\varphi)(a'_0,\dots,a'_n)\\
&\quad = a'_i \cdot m_{A_{s(a'_i)}}\big(a_i \longmapsto {a'}_i^{-1}\cdot \varphi(a'_0,\dots,a'_i\cdot a_i,\dots,a'_n)\big)\\
&\quad = a'_i \cdot ({b'}_i^{-1}\cdot a'_i)^{-1}\cdot m_{A_{s(b'_i)}}\big(\\
&\qquad\qquad a_i \longmapsto ({b'}_i^{-1}\cdot a'_i)\cdot {a'}_i^{-1}\cdot \varphi(a'_0,\dots,a'_i\cdot ({b'}_i^{-1}\cdot a'_i)^{-1}a_i,\dots,a'_n)\big)\\
&\quad = b'_i \cdot m_{A_{s(b'_i)}}\big(a_i \longmapsto {b'}_i^{-1}\cdot \varphi(a'_0,\dots,b'_i\cdot a_i,\dots,a'_n)\big)\\
&\quad = \Phi^n_i(\varphi)(a'_0,\dots,b'_i,\dots,a'_n)
\end{aligned}$$

Hence $\Phi^n_i(\varphi)(a'_0,\dots,a'_n)$ does not depend on a'_i. Therefore, $A^n_{\mathcal{A}}(\varphi\circ L_n(i))$ is constant for all $\varphi \in B(L_n(\mathcal{G}),V')$ and in particular invariant under permutations of the arguments. Thus, $B(L_n(i),V')\circ \mathrm{Alt}^n_{V'}\circ A^n_{\mathcal{A}} = 0$ □

Corollary 4.2.4. Let $\mathcal{G}$ be an amenable groupoid and V a Banach $\mathcal{G}$-module. Then $H^n_b(\mathcal{G},V') = 0$ for all $n \in \mathbb{N}_{\geq 1}$.

Proof. Since the composition $\mathrm{Alt}^*_{V'}\circ A^*_{\mathcal{G}}\colon B(L_n(\mathcal{G}),V') \longrightarrow B(L_*(\mathcal{G}),V')$ is a $\mathcal{G}$-map extending $\mathrm{id}_{V'}$, by the fundamental lemma, Theorem 3.4.7, it induces the identity on bounded cohomology. On the other hand, by Proposition 4.2.3, it factors through the trivial complex $K^*_L(\mathcal{G},\mathcal{G};V')$ in degree greater or equal 1 and hence $H^*_b(\mathcal{G};V') = 0$ for all $n \in \mathbb{N}_{\geq 1}$. □

Corollary 4.2.5 (Algebraic Mapping Theorem)**.** Let $i\colon \mathcal{A} \hookrightarrow \mathcal{G}$ be a pair of groupoids such that $\mathcal{A}$ is amenable. Let V be Banach $\mathcal{G}$-module. Then

$$H^n(j^*)\colon H^*_b(\mathcal{G},\mathcal{A};V') \longrightarrow H^*_b(\mathcal{G};V')$$

is an isometric isomorphism for each $n \in \mathbb{N}_{\geq 2}$.

Proof. The map j^* is norm non-increasing and by Corollary 4.2.4 and the long exaxt sequence, induces a norm non-increasing isomorphism in bounded cohomology in degree greater or equal 2. As before, since the composition $\mathrm{Alt}^*_{V'}\circ A^*_{\mathcal{A}}\colon B(L_*(\mathcal{G}),V') \longrightarrow B(L_*(\mathcal{G}),V')$ is a $\mathcal{G}$-map extending $\mathrm{id}_{V'}$, by the fundamental lemma, Theorem 3.4.7, it induces the identity on bounded cohomology. The map $B(L_n(\mathcal{G}),V') \longrightarrow K^n_L(\mathcal{G},\mathcal{A},V')$ induced by $\mathrm{Alt}^*_{V'}\circ A^*_{\mathcal{A}}$ in degree greater or equal 1 is also norm non-increasing. Therefore, by Proposition 4.2.3, $H^n(j^*)$ is an isometric isomorphism for $n \in \mathbb{N}_{\geq 2}$. □

Remark 4.2.6. Even when considering only groups, the algebraic mapping theorem is wrong in general if we use coefficients in general modules, not just dual spaces. More precisely, Noskov shows that for any Banach V with an isometric $\mathbb{Z}$-action, there is an isomorphism

$$H_b^1(\mathbb{Z};V) \cong \Big\{ v \in V \;\Big|\; \sup_{n\in\mathbb{N}} \Big\| \sum_{i=0}^{n} T^i \cdot v \Big\| < \infty \Big\} \Big/ (T-1)\cdot V,$$

where T denotes a generator of $\mathbb{Z}$, and he gives several examples where the right-hand side is infinite dimensional [65, Section 7.2 and 7.5].

As in the group case, also the converse to Corollary 4.2.4 holds in the strong sense that vanishing in degree 1 for a certain dual space is sufficient for a groupoid to be amenable. Namely, consider the Banach quotient $\Sigma\mathbb{R}[\mathcal{G}] := \operatorname{coker} c_{\mathbb{R}[\mathcal{G}]}$:

Proposition 4.2.7. *Let $\mathcal{G}$ be a groupoid. If $H_b^1(\mathcal{G};(\Sigma\mathbb{R}[\mathcal{G}])') = 0$, then $\mathcal{G}$ is amenable.*

Proof. The proof is basically the same as Noskov's proof in the group case [65, Section 7.1]. By Proposition 3.3.6, we notice that the canonical projection map $\pi\colon B(\mathcal{G},\mathbb{R}[\mathcal{G}]) \longrightarrow \Sigma\mathbb{R}[\mathcal{G}]$ induces a $\mathcal{G}$-isomorphism between $(\Sigma\mathbb{R}[\mathcal{G}])'$ and the $\mathcal{G}$-submodule V of $B(\mathcal{G},\mathbb{R}[\mathcal{G}])'$ of functions that vanish on $\mathbb{R}[\mathcal{G}] = \operatorname{im} c_{\mathbb{R}[\mathcal{G}]}$, and we will identify $(\Sigma\mathbb{R}[\mathcal{G}])'$ with V. Consider the $\mathbb{R}$-+ morphism

$$\begin{aligned} \mu\colon B(\mathcal{G},\mathbb{R}[\mathcal{G}]) &\longrightarrow \mathbb{R}[\mathcal{G}] \\ B(\mathcal{G}_e,\mathbb{R}[G]_e) \ni \varphi &\longmapsto \varphi(\mathrm{id}_e). \end{aligned}$$

We define a bounded $\mathcal{G}$-map

$$\begin{aligned} f\colon C_1(\mathcal{G}) &\longrightarrow V \\ (g_0,g_1) &\longmapsto g_0\cdot g_1\cdot\mu - g_0\cdot\mu. \end{aligned}$$

The map f represents a cocyle in $C_b^1(G;V)$, hence by assumption there is a cochain $b \in C_b^0(\mathcal{G};V)$ such that $f = \delta^0 b$. Set $\nu := (b(\mathrm{id}_e))_{e\in\operatorname{ob}\mathcal{G}} \in V$. Then $\mu - \nu \in B(\mathcal{G},\mathbb{R}[\mathcal{G}])'$ is a $\mathcal{G}$-invariant element in $B(\mathcal{G},\mathbb{R}[\mathcal{G}])'$, because for all $g \in \mathcal{G}$

$$\begin{aligned} g\cdot(\mu-\nu) - (\mu-\nu) &= (g\cdot\mu - \mu) - (g\cdot\nu - \nu) \\ &= f(g,\mathrm{id}_{s(g)}) - \delta^0 b(g,\mathrm{id}_{s(g)}) \\ &= 0. \end{aligned}$$

Furthermore, for any constant function $a \in \mathbb{R}[\mathcal{G}] \subset B(\mathcal{G},\mathbb{R}[\mathcal{G}])$

$$(\mu-\nu)(a) = \mu(a) - \nu(a) = \mu(a) = a.$$

Now, in order to see that one can find an invariant mean, one proceeds as in the proof of [69, Proposition 2.2]. Here, the idea is that after appropriately translating the question in terms of measures, one can assign $\mu-\nu$ an invariant measure and take the positive variation of this measure, using the Jordan decomposition, to find a positive measure corresponding conversely to an invariant mean. □

Chapter 5

Topology and Bounded Groupoid Cohomology

In this chapter, we will discuss bounded cohomology of (pairs of) topological spaces and relate it to the bounded cohomology of the corresponding (pairs of) fundamental groupoids.

In the first section, we associate to a topological space X a $\pi_1(X)$-chain complex $C_*(X)$, which will play a role analogous to the $\pi_1(X, x)$-chain complex $C_*^{\text{sing}}(\widetilde{X})$ in the group setting.

In the second section, we use $C_*(X)$ to define bounded cohomology of a space with twisted coefficients, and show that for connected spaces it coincides with the classical definition. We illustrate how to translate (co-)chain contractions from the group setting into the groupoid setting and deduce that $B(C_*(X), V')$ is a strong resolution of V' for any Banach $\pi_1(X)$-module V'. We also show that this resolution is relatively injective and proof the absolute mapping theorem for groupoids, saying that the bounded cohomology of a space is isometrically isomorphic to the bounded cohomology of its fundamental groupoid.

In the third section, we define bounded cohomology with twisted coefficients for pairs of topological spaces. Finally, we prove the relative mapping theorem.

5.1 The Topological Resolution

In this section, let X be a CW-complex. Let $p\colon \widetilde{X} \longrightarrow X$ be the universal cover. For each $n \in \mathbb{N}$, write $S_n(\widetilde{X})$ for the set of singular n-simplices in $\widetilde{X}$. The group $\text{Deck}(X)$ of deck transformations of p acts (from the left) on $S_n(\widetilde{X})$, $C_n^{\text{sing}}(\widetilde{X})$ and $\widetilde{X}$ as usual. Again, we will denote the fundamental groupoid of X by $\pi_1(X)$. For each $x \in \widetilde{X}$, denote the connected component of $\widetilde{X}$ containing x by $\widetilde{X}_x$. For a point $x \in X$, we will write $\widetilde{x}$ for a choice of a lift of x, and similarly $\widetilde{\sigma}$ for a choice of a lift of a simplex $\sigma \in S_n(X)$. By a slight abuse of notation, if $\sigma \in S_n(X)$, we will also write $\sigma(0), \ldots, \sigma(n)$ to denote the vertices of σ.

Remark 5.1.1. In order to introduce bounded cohomology of a space X with twisted coefficients, we will need the existence of a universal cover of X. In the following sections, we will therefore restrict to CW-complexes, though the results will also hold for the class of topological spaces admitting a universal cover.

5.1.1 Definition of the Topological Resolution

Definition 5.1.2. Let X be a CW-complex. There is a partial action of $\pi_1(X)$ on $\widetilde{X}$, i.e., for each $\gamma \in \pi_1(X)$ there is a bijection

$$\begin{aligned} \rho_\gamma \colon p^{-1}(s(\gamma)) &\longrightarrow p^{-1}(t(\gamma)) \\ x &\longmapsto \gamma \cdot x := \widetilde{\gamma_x}(1). \end{aligned}$$

Here, for each $x \in \widetilde{X}$, let $\widetilde{\gamma_x}$ denote the lift of a representative of γ starting at x. Furthermore, we have for all $x \in X$

$$\rho_{\mathrm{id}_x} = \mathrm{id}_{p^{-1}(x)}$$

and for all $\gamma, \gamma' \in \pi_1(X)$ with $s(\gamma') = t(\gamma)$

$$\rho_{\gamma' \cdot \gamma} = \rho_{\gamma'} \circ \rho_\gamma.$$

This induces a $\pi_1(X)$-module structure on $\mathbb{R}[\widetilde{X}] := \bigoplus_{\widetilde{X}} \mathbb{R}$.

Similarly, for all $\emptyset \neq I \subset X$ this also induces a $\pi_1(X, I)$-module structure on $\mathbb{R}[\widetilde{I}]$, where $\widetilde{I} := p^{-1}(I)$.

Lemma 5.1.3. For all $x \in \widetilde{X}$, all $\gamma \in \pi_1(X)$ with $s(\gamma) = p(x)$ and $\beta \in \mathrm{Deck}(X)$ we have

$$\beta(\gamma \cdot x) = \gamma \cdot \beta(x).$$

In other words, $\mathbb{R}[\widetilde{X}]$ is a $(\mathrm{Deck}(X), \pi_1(X))$-bi-module, i.e., represented by a functor $\pi_1(X) \longrightarrow \mathrm{Deck}(X)\text{-}\mathsf{Mod}$.

Proof. If $\widetilde{\gamma_x}$ is a lift of a representative of γ starting in x, then $\beta \cdot \widetilde{\gamma_x}$ is a lift of a representative of γ starting in $\beta(x)$ hence

$$\begin{aligned} \gamma \cdot (\beta(x)) &= \widetilde{\gamma_{\beta(x)}}(1) \\ &= \beta(\widetilde{\gamma_x}(1)) \\ &= \beta(\gamma \cdot x). \end{aligned}$$ ☐

Now, we introduce a $\pi_1(X)$-version of the $\pi_1(X, x)$-chain complex $C_*^{\mathrm{sing}}(\widetilde{X})$:

Definition 5.1.4.

(i) For all $n \in \mathbb{N}$, we define the set

$$Q_n(X) := \frac{\{(\sigma, x) \in S_n(\widetilde{X}) \times \widetilde{X} \mid \operatorname{im} \sigma \subset \widetilde{X}_x\}}{\Big((\beta \cdot \sigma, \beta \cdot x) \sim (\sigma, x) \mid \beta \in \mathrm{Deck}(X)\Big)}.$$

By a slight abuse of notation, we denote the class of a $(\sigma, x) \in S_n(\widetilde{X}) \times \widetilde{X}$ in $Q_n(X)$ also by (σ, x). For all $e \in X$ we set

$$Q_n(X)_e := \{(\sigma, x) \in Q_n(X) \mid p(x) = e\}.$$

(ii) For all $n \in \mathbb{N}$ we define a $\pi_1(X)$-module $C_n(X)$ via

$$C_n(X) = (\mathbb{R}\langle Q_n(X)_e\rangle)_{e\in X}$$

with the partial $\pi_1(X)$-action given by setting for each $\gamma \in \pi_1(X)$

$$\begin{aligned} \rho_\gamma \colon C_n(X)_{s(\gamma)} &\longrightarrow C_n(X)_{t(\gamma)} \\ (\sigma, x) &\longmapsto (\sigma, \gamma \cdot x). \end{aligned}$$

This action is well-defined by Lemma 5.1.3.

(iii) For all $n \in \mathbb{N}_{>0}$, we define boundary maps $\partial_n \colon C_n(X) \longrightarrow C_{n-1}(X)$ via

$$\partial_n(\sigma, x) := \sum_{i=0}^{n} (-1)^i (\partial_{i,n}\sigma, x).$$

These maps are clearly $\pi_1(X)$-equivariant and well-defined since the action by deck transformations is compatible with the usual boundary maps. Furthermore, these are obviously boundary maps.

(iv) Additionally, we will consider the canonical augmentation map

$$\begin{aligned} \varepsilon \colon C_0(X) &\longrightarrow \mathbb{R}[X] \\ (\sigma, x) &\longmapsto p(x) \cdot 1. \end{aligned}$$

Here, we denote by $\mathbb{R}[X]$ the trivial $\pi_1(X)$-module $\mathbb{R}[\pi_1(X)]$.

(v) We also endow $C_*(X)$ with the ℓ^1-norm with respect to the basis $Q_*(X)$. Then $C_n(X)$ is a normed $\pi_1(X)$-module for all $n \in \mathbb{N}$ and $\|\partial_n\|_\infty \le n+1$.

(vi) For any subset $\emptyset \neq I \subset X$, we analogously define a normed $\pi_1(X, I)$-chain complex $(C_*(X : I))_{*\in\mathbb{N}}$.

Remark 5.1.5. We can make this definition more concise using tensor products over group modules and the $(\mathrm{Deck}(X), \pi_1(X))$-bi-module structure on $\mathbb{R}[\widetilde{X}]$. Then $C_n(X)$ is nothing else than $C_n^{\mathrm{sing}}(\widetilde{X}) \otimes_{\mathrm{Deck}(X)} \mathbb{R}[\widetilde{X}]$, i.e., given by the composition

$$\pi_1(X) \xrightarrow{\mathbb{R}[\widetilde{X}]} \mathrm{Deck}(X)\text{-}\mathsf{Mod} \xrightarrow{\otimes_{\mathrm{Deck}(X)} C_n^{\mathrm{sing}}(\widetilde{X})} \mathbb{R}\text{-}\mathsf{Mod}\,.$$

Remark 5.1.6. Let X be a connected topological space and $x \in X$ a point. Then $C_*(X : \{x\}) = C_*^{\mathrm{sing}}(\widetilde{X}; \mathbb{R})$ as $\pi_1(X, x)$-modules. Thus the definition, and the ones that will follow, generalise the classical situation.

Proposition 5.1.7 (Functoriality). *Let $f \colon X \longrightarrow Y$ be a continuous map between two CW-complexes. Let $\widetilde{f} \colon \widetilde{X} \longrightarrow \widetilde{Y}$ be a lift. Then, $\widetilde{f}$ is $\pi_1(X)$-equivariant, i.e., for all $\gamma \in \pi_1(X)$ and all $x \in p_X^{-1}(s(\gamma))$ we have*

$$\widetilde{f}(\gamma \cdot x) = \pi_1(f)(\gamma) \cdot \widetilde{f}(x).$$

Therefore, the following map is a $\pi_1(X)$-chain map with respect to the induced $\pi_1(X)$-structure on $C_*(Y)$.

$$\begin{aligned} C_*(f)\colon C_*(X) &\longrightarrow C_*(Y) \\ (\sigma, x) &\longmapsto (\widetilde{f} \circ \sigma, \widetilde{f}(x)). \end{aligned}$$

The map $C_*(f)$ does not depend on the choice of the lift $\widetilde{f}$. In particular, C_* is functorial in the sense that $C_*(g \circ f) = (\pi_1(f)^* C_*(g)) \circ C_*(f)$ for all continuous maps $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$.

Proof. Let $\widetilde{\gamma}_x$ be a lift of γ starting at x. Then

$$p_Y \circ \widetilde{f} \circ \widetilde{\gamma}_x = f \circ p_X \circ \widetilde{\gamma}_x = f \circ \gamma = \pi_1(f)(\gamma).$$

Hence $\widetilde{f} \circ \widetilde{\gamma}_x$ is a lift of $\pi_1(f)(\gamma)$ starting at $\widetilde{f}(x)$ and therefore

$$\widetilde{f}(\gamma \cdot x) = \widetilde{f}(\widetilde{\gamma}_x(1)) = (\widetilde{f} \circ \widetilde{\gamma}_x)(1) = \pi_1(f)(\gamma) \cdot \widetilde{f}(x).$$

The lifts of f are exactly given by $\{\beta \circ \widetilde{f} \mid \beta \in \mathrm{Deck}(Y)\}$. For all $\beta \in \mathrm{Deck}(X)$ the map $\widetilde{f} \circ \beta$ is a lift of f, and therefore there exists a $\beta' \in \mathrm{Deck}(Y)$ such that $\beta' \circ \widetilde{f} = \widetilde{f} \circ \beta$, hence for all $(\sigma, x) \in Q_n(X)$

$$(\widetilde{f} \circ (\beta \cdot \sigma), \widetilde{f}(\beta \cdot x)) = (\beta' \cdot (\widetilde{f} \circ \sigma), \beta' \cdot \widetilde{f}(x)) = (\widetilde{f} \circ \sigma, \widetilde{f}(x)).$$

Thus $C_n(f)$ is well-defined. Clearly, for all $\beta \in \mathrm{Deck}(Y)$, the lifts $\widetilde{f}$ and $\beta \cdot \widetilde{f}$ induce the same map $C_n(f)$, hence $C_n(f)$ does not depend on the choice of the lift. Therefore, the construction is functorial since the composition of the lifts of two maps is a lift of the composition of the two maps. The family $C_*(f)$ is a chain map since $C_*^{\mathrm{sing}}(\widetilde{f})$ is a chain map. Furthermore, by definition it is compatible with the induced $\pi_1(X)$-structure on $C_*(Y)$. □

Remark 5.1.8. Starting from this construction, one can define (co-)homology, ℓ^1-homology and bounded cohomology of a space X with twisted coefficients in a $\pi_1(X)$-module just as in Chapter 3. We will restrict our attention to bounded cohomology here, though.

5.2 Bounded Cohomology of Topological Spaces

In this section, we use the $\pi_1(X)$-chain complex $C_*(X)$ to define bounded cohomology of X with twisted coefficients in a Banach $\pi_1(X)$-module V, slightly extending the usual definition of bounded cohomology with twisted coefficients. We will then study the normed cochain complex $B(C_*(X), V')$, show that it is a strong, relatively injective $\pi_1(X)$-resolution of V' and derive the absolute mapping theorem for groupoids, i.e., that the bounded cohomology of X is isometrically isomorphic to the bounded cohomology of $\pi_1(X)$, preparing the ground for our proof of the relative mapping theorem in the next section.

5.2.1 Bounded Cohomology of Topological Spaces

We will now define bounded cohomology of topological spaces X with twisted coefficients in a Banach $\pi_1(X)$-module, and show that this definition coincides with the usual definition of bounded cohomology of a topological space whenever bounded cohomology in the usual definition is defined.

Similar to the case of bounded cohomology of a groupoid, we begin by defining the domain category for bounded cohomology of a space:

Definition 5.2.1 (Domain categories for bounded cohomology of spaces)**.** We define a category $\mathsf{Top}\overline{\mathsf{Ban}}$ by setting

(i) Objects in $\mathsf{Top}\overline{\mathsf{Ban}}$ are pairs (X, V), where X is a topological space and V is a Banach $\pi_1(X)$-module.

(ii) A morphism $(X, V) \longrightarrow (Y, W)$ in the category $\mathsf{Top}\overline{\mathsf{Ban}}$ is a pair (f, φ), where $f\colon X \longrightarrow Y$ is a continuous map and $\varphi\colon \pi_1(f)^*W \longrightarrow V$ is a bounded $\pi_1(X)$-map.

(iii) We define the composition in $\mathsf{Top}\overline{\mathsf{Ban}}$ as follows: For each pair of morphisms $(f, \varphi)\colon (X, U) \longrightarrow (Y, V)$, $(g, \psi)\colon (Y, V) \longrightarrow (Z, W)$ set

$$(g, \psi) \circ (f, \varphi) := (g \circ f, \varphi \circ (\pi_1(f)^*\psi)).$$

Definition 5.2.2 (The Banach Bar Complex with coefficients)**.** Let X be a CW-complex and V a Banach $\pi_1(X)$-module. Then:

(i) We write

$$C_b^*(X; V) := B_{\pi_1(X)}(C_*(X), V).$$

Together with $\|\cdot\|_\infty$, this is a normed $\mathbb{R}$-cochain complex.

(ii) If $(f, \varphi)\colon (X, V) \longrightarrow (Y, W)$ is a morphism in $\mathsf{Top}\overline{\mathsf{Ban}}$, we write $C_b^*(f, \varphi)$ for the $\mathbb{R}$-cochain map

$$\begin{aligned} C_b^*(Y; W) &\longrightarrow C_b^*(X; V) \\ \alpha &\longmapsto \varphi \circ (\pi_1(f)^*\alpha) \circ C_*(f). \end{aligned}$$

This defines a contravariant functor $C_b^*\colon \mathsf{Top}\overline{\mathsf{Ban}} \longrightarrow {}_{\mathbb{R}}\mathsf{Ch}^{\|\cdot\|}$.

Definition 5.2.3. Let X be a CW-complex, V a Banach $\pi_1(X)$-module. We call the cohomology

$$H_b^*(X; V) := H^*(C_b^*(X; V)),$$

together with the induced semi-norm on $H_b^*(X; V)$, the *bounded cohomology of X with twisted coefficients in V.*

This defines a contravariant functor $H_b^*\colon \mathsf{Top}\overline{\mathsf{Ban}} \longrightarrow \mathbb{R}\text{-}\mathsf{Mod}_*^{\|\cdot\|}$.

Bounded cohomology with twisted coefficients is normally defined only for connected spaces and for general spaces only with trivial coefficients. We will now show that our definition coincides in this cases with the classical one.

Lemma 5.2.4. Let X be a connected CW-complex, $x \in X$ and V a Banach $\pi_1(X)$-module. Then there is a canonical isometric $\mathbb{R}$-cochain isomorphism

$$\begin{aligned} S^*_{X,V} \colon C^*_b(X;V) &\longrightarrow C^*_b(X:\{x\};V_x) \\ (\varphi_e)_{e\in X} &\longmapsto \varphi_x. \end{aligned}$$

This is natural in the sense that for each $(f,\psi)\colon (X,V) \longrightarrow (Y,W)$ in $\mathsf{Top}\overline{\mathsf{Ban}}$, such that X and Y are connected, and each $x \in X$ the following diagram commutes

$$\begin{tikzcd} C^*_b(Y;W) \arrow[r, "S^*_{Y,W}"] \arrow[d, "C^*_b(f,\psi)"] & C^*_b(Y:\{f(x)\};W_{f(x)}) \arrow[d, "C^*_b(f,\psi_x)"] \\ C^*_b(X;V) \arrow[r, "S^*_{X,V}"] & C^*_b(X:\{x\};V_x). \end{tikzcd}$$

Proof. Because $\pi_1(X)$ is connected, the map $e \longmapsto \|\varphi_e\|_\infty$ is constant for all $(\varphi_e)_{e\in X} \in C^*_b(X;V)$, hence the map $S^*_{X,V}\colon (\varphi_e)_{e\in X} \longmapsto \varphi_x$ is isometric and injective. It is a cochain map, since the coboundary operators are defined pointwise. An inverse is given by choosing for each $y \in X$ a path $\gamma_y \in \mathrm{Mor}_{\pi_1(X)}(x,y)$ from x to y and setting

$$\begin{aligned} R^* \colon C^*_b(X:\{x\};V_x) &\longrightarrow C^*_b(X;V) \\ \varphi &\longmapsto (\gamma_y \cdot \varphi)_{y\in X}. \end{aligned}$$

The map R^* does not depend on the choice of $(\gamma_y)_{y\in X}$ since by the $\pi_1(X,x)$-invariance of φ, we have for all $\gamma \in \mathrm{Mor}_{\pi_1(X)}(x,y)$

$$\gamma \cdot \varphi = \gamma_y \cdot (\gamma_y^{-1} \cdot \gamma) \cdot \varphi = \gamma_y \cdot \varphi.$$

In particular, for all $\varphi \in C^*_b(X:\{x\};V_x)$ the map $R^*(\varphi)$ is $\pi_1(X)$-invariant and clearly bounded. ☐

Similar to Proposition 3.3.18, we see that bounded cohomology is finitely additive:

Proposition 5.2.5 (Bounded cohomology and disjoint unions of spaces)**.** *Let X be a CW-complex with finitely many connected components and let V be a Banach $\pi_1(X)$-module. Let $X = \amalg_{\lambda\in\Lambda} X^\lambda$ be the partition in connected components. For each $\lambda \in \Lambda$, write V^λ for the $\pi_1(X^\lambda)$-module structure on V induced by the inclusion $\pi_1(X^\lambda) \hookrightarrow \pi_1(X)$. Then the family $(\pi_1(X^\lambda) \hookrightarrow \pi_1(X))_{\lambda\in\Lambda}$ of injective groupoid maps induces an isometric isomorphism*

$$H^*_b(X;V) \longrightarrow \prod_{\lambda\in\Lambda} H^*_b(X^\lambda;V^\lambda)$$

with respect to the product semi-norm, see Definition 3.3.17. Composing with the isometric isomorphisms given by Lemma 5.2.4, we get an isometric isomorphism

$$H^*_b(X;V) \longrightarrow \prod_{\lambda\in\Lambda} H^*_b(X^\lambda:\{x(\lambda)\};V_{x(\lambda)})$$

for any choice of points $x(\lambda) \in X^\lambda$.

Proof. The partition of X into connected components induces a decomposition of $S_n(\widetilde{X})$, $\widetilde{X}$ and $C_*(X)$, which is compatible with the boundary operators and the $\pi_1(X)$-action and preserved under applying $B_{\pi_1(X)}(\,\cdot\,,V)$. The induced semi-norm is exactly the product semi-norm. □

Proposition 5.2.6. *Let X be a CW-complex. The following map is an isometric $\mathbb{R}$-cochain isomorphism, natural in X*

$$\begin{aligned} R^*\colon B(C_*^{\mathrm{sing}}(X;\mathbb{R}),\mathbb{R}) &\longrightarrow C_b^*(X;\mathbb{R}[X]) \\ \varphi &\longmapsto \Big((\sigma,x)\longmapsto \varphi(p\circ\sigma)\Big). \end{aligned}$$

Proof. Clearly, for all $n\in\mathbb{N}$ and all $\varphi\in B(C_n^{\mathrm{sing}}(X;\mathbb{R}),\mathbb{R})$, the map $R^n(\varphi)$ is well-defined, $\pi_1(X)$-invariant and

$$\|R^n(\varphi)\|_\infty = \sup_{(\sigma,x)\in Q_n(X)} |\varphi(p\circ\sigma)| = \sup_{\sigma'\in S_n(X)} |\varphi(\sigma')| = \|\varphi\|_\infty.$$

Thus, $R^n(\varphi)$ is bounded and R^n an isometry. The family R^* is also clearly an $\mathbb{R}$-cochain map. An inverse to R^* is given by

$$\begin{aligned} S^*\colon C_b^*(X;\overline{\mathbb{R}}) &\longrightarrow B(C_*^{\mathrm{sing}}(X;\mathbb{R}),\mathbb{R}) \\ \psi &\longmapsto \big(\sigma\longmapsto \psi_{\sigma(0)}(\widetilde{\sigma},\widetilde{\sigma}(0))\big), \end{aligned}$$

where $\widetilde{\sigma}$ denotes a lift of σ. A short calculation shows, that this is indeed a well-defined inverse to R^*. □

Remark 5.2.7. Gromov [45] (and Kim and Kuessner [51]) studies bounded cohomology of topological spaces also via so-called multicomplexes. He shows that the bounded cohomology of a space X can be calculated using the geometric realisation of a multicomplex $K(X)$ associated to X. This multicomplex $K(X)$ can be defined as follows: The 0-skeleton are the points in X, the 1-skeleton is given by picking a representative of each homotopy class relative endpoints of paths in X with distinct start and endpoint. Inductively, the n-skeleton is build by picking a representative of each homotopy class of singular n-simplices with distinct vertex points relative to the boundary, in a fashion compatible with the choices for the $(n-1)$-skeleton. Finally, one identifies simplices if they have the same 1-skeleton.

Also $P_*(\pi_1(X))$ (and $\pi_1(X)$) is defined in terms of homotopy classes of paths, though we consider all paths, not just paths with distinct start and endpoint. Furthermore, the higher-dimensional simplices in $P_*(\pi_1(X))$ are not directly related to singular simplices. In any case, the methods we use a very different from Gromov's and more in the tradition of Ivanov's approach.

5.2.2 The Absolute Mapping Theorem

In this section, we will give a proof of the absolute mapping theorem for groupoids, i.e., we will show that there is an isometric isomorphism between the bounded cohomology of a CW-complex and the bounded cohomology of its fundamental groupoid. This will be done by translating Ivanov's proof [48] of the theorem for groups into the groupoid setting.

Let X be a CW-complex and V a Banach $\pi_1(X)$-module. We will study the Banach $\pi_1(X)$-cochain complex $B(C_*(X), V)$ together with the canonical $\pi_1(X)$-augmentation map

$$\begin{aligned} \nu\colon V &\longrightarrow B(C_0(X), V) \\ v &\longmapsto ((\sigma, x) \longmapsto v). \end{aligned}$$

We will show that $(B(C_*(X), V), \nu)$ is a strong, relatively injective resolution of V, and then deduce the mapping theorem from the fundamental lemma. In order to prove that this is a strong resolution, we will first discuss how to translate (co-)chain contractions from the group setting into the groupoid world.

We will start with the homological situation. Recall that for all $x \in \widetilde{X}$ there is a canonical augmentation map

$$\begin{aligned} \varepsilon^x\colon C_0^{\mathrm{sing}}(\widetilde{X}_x; \mathbb{R}) &\longrightarrow \mathbb{R} \\ \sigma &\longmapsto 1. \end{aligned}$$

Definition 5.2.8 (Pointed equivariant chain contractions)**.** Let X be a CW-complex. We call a family

$$\Bigl(\bigl(s_*^x\colon C_*^{\mathrm{sing}}(\widetilde{X}_x; \mathbb{R}) \longrightarrow C_{*+1}^{\mathrm{sing}}(\widetilde{X}_x; \mathbb{R})\bigr)_{*\in\mathbb{N}},\ s_{-1}^x\colon \mathbb{R} \longrightarrow C_0^{\mathrm{sing}}(\widetilde{X}_x; \mathbb{R})\Bigr)_{x\in\widetilde{X}}$$

of chain contractions of the augmented chain complexes $(C_*^{\mathrm{sing}}(\widetilde{X}_x; \mathbb{R}), \varepsilon^x)_{x\in\widetilde{X}}$ *pointed equivariant over* X if

(i) The family is $\mathrm{Deck}(X)$-equivariant, i.e., for all $x \in \widetilde{X}$, all $\sigma \in S_*(\widetilde{X}_x)$ and all $\beta \in \mathrm{Deck}(X)$
$$s_*^{\beta\cdot x}(\beta \cdot \sigma) = \beta \cdot s_*^x(\sigma).$$

(ii) The contractions are pointed, i.e., for all $x \in \widetilde{X}$
$$s_{-1}^x(1) = x.$$

For the cohomological version, recall that for any $\mathbb{R}$-module V and any $x \in \widetilde{X}$, there is a canonical augmentation map

$$\begin{aligned} \nu_x\colon V &\longrightarrow B(C_0^{\mathrm{sing}}(\widetilde{X}_x; \mathbb{R}), V) \\ v &\longmapsto \bigl(\sigma \longmapsto v\bigr). \end{aligned}$$

Now, we dually define pointed equivariant families of cochain contractions in the bounded setting:

Definition 5.2.9. Let X be a CW-complex. Let $(V_x)_{x\in X}$ be a family of Banach $\mathbb{R}$-modules. We call a family of cochain contractions

$$\begin{pmatrix} \bigl(s_x^*\colon B(C_*^{\mathrm{sing}}(\widetilde{X}_x; \mathbb{R}), V_{p(x)}) \longrightarrow B(C_{*-1}^{\mathrm{sing}}(\widetilde{X}_x; \mathbb{R}), V_{p(x)})\bigr)_{*\in\mathbb{N}_{>0}} \\ s_x^0\colon B(C_0^{\mathrm{sing}}(\widetilde{X}_x; \mathbb{R}), V_{p(x)}) \longrightarrow V_{p(x)} \end{pmatrix}_{x\in\widetilde{X}}$$

of the family of augmented cochain complexes $(B(C_*^{\mathrm{sing}}(\widetilde{X}_x; \mathbb{R}), V_{p(x)}), \nu_x)_{x\in\widetilde{X}}$ *pointed equivariant over* (X, V) if

(i) The family is $\mathrm{Deck}(X)$-equivariant, i.e., for all $\beta \in \mathrm{Deck}(X)$, all $x \in \widetilde{X}$ and all $\varphi \in B(C_*^{\mathrm{sing}}(\widetilde{X}_x), V_{p(x)})$

$$s^*_{\beta \cdot x}(\beta * \varphi) = \beta * s^*_x(\varphi).$$

Here, we write $*$ to denote the action of $\mathrm{Deck}(X)$ on the cochain complex $B(C_*^{\mathrm{sing}}(\widetilde{X}_x; \mathbb{R}), V_{p(x)})$ given by endowing the module $V_{p(x)}$ with the trivial $\mathrm{Deck}(X)$-action.

(ii) They are pointed, i.e., for all $x \in \widetilde{X}$ and all $\varphi \in B(C_0^{\mathrm{sing}}(\widetilde{X}_x), V_{p(x)})$

$$s^0_x(\varphi) = \varphi(x).$$

By a short calculation we see that the definition of pointed equivariant cochain contractions is indeed dual to the definition of pointed equivariant chain contractions:

Lemma 5.2.10. Let X be a CW-complex and let $(s^x_*)_{x \in \widetilde{X}}$ be a pointed equivariant family of bounded chain contractions over X. Let $(V_x)_{x \in X}$ be a family of Banach $\mathbb{R}$-modules. Then $B(s^x_*, V_{p(x)})_{x \in \widetilde{X}}$ is a pointed equivariant family of cochain contractions over (X, V).

Remark 5.2.11. Of course, given a family of pointed (co-)chain contractions, one can always find a pointed equivariant family by choosing one contraction for a point in each fibre and defining the other contractions by translation with deck transformations, i.e.: If $(s^*_x)_{x \in \widetilde{X}}$ is a family of pointed cochain contractions, choose a lift $\widetilde{x_0}$ for each $x_0 \in X$ and set for each $\beta \in \mathrm{Deck}(X)$ and each $\varphi \in B(C_*^{\mathrm{sing}}(\widetilde{X}_x), V_{p(x)})$

$$t^*_{\beta \cdot \widetilde{x}_0}(\varphi) = \beta * s^*_{\widetilde{x}_0}(\beta^{-1} * \varphi).$$

Then $(t^*_x)_{x \in \widetilde{X}}$ is a pointed equivariant family of cochain contractions.

Example 5.2.12. Assume X to be aspherical (but not necessarily connected), i.e., assume that the higher homotopy groups of all connected components of X vanish. Then the space $\widetilde{X}_x$ is contractible for each $x \in \widetilde{X}$ and a pointed chain contraction is given by coning with respect to x, see Appendix A. Hence by the remark there exists a pointed equivariant family of chain contractions over X.

Proposition 5.2.13. *Let X be a CW-complex. Let $(s^x_*)_{x \in \widetilde{X}}$ be a pointed equivariant family of chain contractions over X. Then the maps*

$$\begin{aligned} \forall_{n \in \mathbb{N}} \quad s_n \colon C_n(X) &\longrightarrow C_{n+1}(X) \\ (\sigma, x) &\longmapsto (s^x_n(\sigma), x) \\ s_{-1} \colon \mathbb{R}[X)] &\longrightarrow C_0(X) \\ e &\longmapsto (\widetilde{e}, \widetilde{e}) \end{aligned}$$

define a chain contraction of the augmented cochain complex $(C_*(X), \varepsilon)$.

Proof.

- Well-defined: For all $n \in \mathbb{N}$, $(\sigma, x) \in Q_n(X)$ and $\beta \in \mathrm{Deck}(X)$

$$s_n(\beta \cdot \sigma, \beta \cdot x) = (s_n^{\beta \cdot x}(\beta \cdot \sigma), \beta \cdot x) = (\beta \cdot s^x(\sigma), \beta \cdot x) = (s^x_n(\sigma), x).$$

- A chain contraction: For all $n \in \mathbb{N}_{>0}$ and all $(\sigma, x) \in Q_n(X)$:

  $$(\partial_{n+1} \circ s_n + s_{n-1} \circ \partial_n)(\sigma, x) = ((\partial_{n+1} s_n^x + s_{n-1}^x \partial_n)(\sigma), x) = (\sigma, x).$$

  For all $(\sigma, x) \in Q_0(X)$:

  $$\begin{aligned}(\partial_1 s_0 + s_{-1}\varepsilon)(\sigma, x) &= (\partial_1 s_0^x(\sigma), x) + s_{-1}(p(x)) \\ &= (\partial_1 s_0^x(\sigma), x) + (x, x) \\ &= (\partial_1 s_0^x(\sigma) + x, x) \\ &= (\partial_1 s_0^x(\sigma) + s_{-1}^x \varepsilon(\sigma), x) \\ &= (\sigma, x).\end{aligned}$$

  For all $e \in X$:

  $$\varepsilon s_{-1}(1 \cdot e) = \varepsilon(\widetilde{e}, \widetilde{e}) = 1 \cdot e. \qquad \square$$

Lemma 5.2.14. Let X be a CW-complex. For each $x \in \widetilde{X}$, the map

$$\begin{aligned} i_x \colon C_*^{\text{sing}}(\widetilde{X}_x) &\longrightarrow C_*(X)_{p(x)} \\ \sigma &\longmapsto (\sigma, x) \end{aligned}$$

is an isometric chain isomorphism with respect to the restriction of the boundary map to $C_*(X)_{p(x)}$.

Proof. It is a chain map since the boundary operators act only on the first argument. An inverse is given by the map $(\sigma, y) \longmapsto \beta_y(\sigma)$, where $\beta_y \in \operatorname{Deck}(X_{p(x)})$ is the unique element such that $\beta_y \cdot y = x$. $\square$

Proposition 5.2.15. *Let X be a CW-complex and let V be a Banach $\pi_1(X)$-module. Let $(s_x^*)_{x \in \widetilde{x}}$ be a pointed equivariant family of cochain contractions over (X, V). Then*

$$\begin{aligned} \forall_{n \in \mathbb{N}_{>0}} \quad s^n \colon B(C_n(X), V) &\longrightarrow B(C_{n-1}(X), V) \\ \varphi &\longmapsto \Big((\sigma, x) \longmapsto s_x^n(\varphi \circ i_x)(\sigma)\Big) \\ s^0 \colon B(C_0(X), V) &\longrightarrow V \\ B(C_0(X), V)_e \ni \varphi &\longmapsto \varphi(\widetilde{e}, \widetilde{e}) \end{aligned}$$

defines a cochain contraction of $(B(C_*(X), V), \nu)$. If the $(s_x^*)_{x \in \widetilde{X}}$ are strong, so is s^*.

Proof.

- For each $\varphi \in B(C_n(X), V)$, the map $s^n(\varphi)$ is well-defined: We have for all $(\sigma, x) \in Q_n(X)$ and all $\beta \in \operatorname{Deck}(X)$

  $$\begin{aligned} s^n(\varphi)(\beta\sigma, \beta x) &= s_{\beta \cdot x}^n(\varphi \circ i_{\beta \cdot x})(\beta\sigma) \\ &= s_{\beta \cdot x}^n(\beta * (\varphi \circ i_x))(\beta\sigma) \\ &= s^n(\varphi)(\sigma, x). \end{aligned}$$

- This defines a cochain contraction: We have for all $n \in \mathbb{N}_{>0}$, for all $\varphi \in B(C_n(X), V)$ and all $(\sigma, x) \in Q_n(X)$

$$\begin{aligned}
&(s^{n+1} \circ \delta^n + \delta^{n-1} \circ s^n)(\varphi)(\sigma, x) \\
&\quad = s^{n+1}(\delta^n(\varphi))(\sigma, x) + \sum_{i=0}^{n} (-1)^i \cdot s^n(\varphi)(\partial_{i,n}\sigma, x) \\
&\quad = s_x^{n+1}(\delta^n(\varphi) \circ i_x)(\sigma) + \sum_{i=0}^{n} (-1)^i \cdot s_x^n(\varphi \circ i_x)(\partial_{i,n}\sigma) \\
&\quad = s_x^{n+1}(\delta^n(\varphi \circ i_x))(\sigma) + s_x^n(\varphi \circ i_x)(\partial_n \sigma) \\
&\quad = (s_x^{n+1} \circ \delta^n + \delta^{n-1} \circ s_x^n)(\varphi \circ i_x)(\sigma) \\
&\quad = (\varphi \circ i_x)(\sigma) \\
&\quad = \varphi(\sigma, x).
\end{aligned}$$

Furthermore for all $\varphi \in B(C_0(X), V)$ and all $(\sigma, x) \in Q_0(X)$ we have

$$\begin{aligned}
(s^1 \circ \delta^0 + \nu \circ s^0)(\varphi)(\sigma, x) &= s_x^1((\delta^0\varphi) \circ i_x)(\sigma) + s^0(\varphi) \\
&= s_x^1(\delta^0(\varphi \circ i_x))(\sigma) + \varphi(x, x) \\
&= s_x^1(\delta^0(\varphi \circ i_x))(\sigma) + s_x^0(\varphi \circ i_x) \\
&= (s_x^1 \circ \delta^0 + \nu_x \circ s_x^0)(\varphi \circ i_x)(\sigma) \\
&= (\varphi \circ i_x)(\sigma) \\
&= \varphi(\sigma, x).
\end{aligned}$$

- If the $(s_x^*)_{x \in \widetilde{X}}$ are strong, we note that for all $\varphi \in B(C_n(X), V)$

$$\begin{aligned}
\|s^n\varphi\|_\infty &= \sup_{(\sigma,x) \in Q_n(X)} |s^n\varphi(\sigma, x)| \\
&= \sup_{(\sigma,x) \in Q_n(X)} |s_x^n(\varphi \circ i_x)(\sigma)| \\
&\leq \sup_{x \in \widetilde{X}} \|s_x^n(\varphi \circ i_x)\|_\infty \\
&\leq \sup_{x \in \widetilde{X}} \|(\varphi \circ i_x)\|_\infty \\
&\leq \|\varphi\|_\infty.
\end{aligned}$$

□

Theorem 5.2.16 ([48, 22, 54])**.** *Let X be a connected CW-complex and V a Banach $\pi_1(X, y)$-module. Then for each $x \in \widetilde{X}$ there is a strong pointed cochain contraction*

$$(s_x^* \colon B(C_*^{\mathrm{sing}}(\widetilde{X}; \mathbb{R}), V') \longrightarrow B(C_{*-1}^{\mathrm{sing}}(\widetilde{X}; \mathbb{R}), V'))_{* \in \mathbb{N}}$$

Here, V' denotes the topological dual of V.

Sketch of proof. The main step in Ivanov's proof of the absolute mapping theorem in the group setting is the construction of (partial) strong $\mathbb{R}$-cochain contractions for the cochain complex $B(C_*^{\mathrm{sing}}(\widetilde{X}; \mathbb{R}), \mathbb{R})$ for X a countable CW-complex [48, Theorem 2.4]. This was extended by Löh to cochain complexes with twisted coefficients [54, Lemma B2] in a dual space and in each case, the cochain contractions can be chosen to be pointed. As noted by Bühler, the assumption that X is countable is actually not necessary [22]. □

We will discuss the proof of Theorem 5.2.16 in more detail in Appendix A.

We will use the following simple observation to translate between the fundamental groupoid and the universal cover:

Remark 5.2.17. Let $x, y \in \widetilde{X}$ be two points in the same connected component of $\widetilde{X}$. We write $\gamma_{x,y}$ for the unique element in $\mathrm{Mor}_{\pi_1(X)}(p(x), p(y))$, such that $\gamma_{x,y} \cdot x = y$. Geometrically, this is given by the projection to X of any path in $\widetilde{X}$ from x to y. By definition, we have for all $\beta \in \mathrm{Deck}(X)$, all $g \in \pi_1(X)$ with $s(g) = p(y)$ and all $h \in \pi_1(X)$ with $t(h) = p(x)$

$$\begin{aligned}\gamma_{\beta\cdot x,\beta\cdot y} &= \gamma_{x,y}\\ g \cdot \gamma_{x,y} &= \gamma_{x,g\cdot y}\\ \gamma_{x,y} \cdot h &= \gamma_{h^{-1}\cdot x,y}.\end{aligned}$$

Proposition 5.2.18. *Let X be a CW-complex and let V be a Banach $\pi_1(X)$-module. Then for each $n \in \mathbb{N}$, the Banach $\pi_1(X)$-module $B(C_n(X), V)$ is relatively injective.*

Proof. Let $j \colon U \longrightarrow W$ be a relatively injective map between Banach $\pi_1(X)$-modules. Assume that $\tau \colon W \longrightarrow U$ is a splitting of j as in Definition 3.4.1(i). Let

$$\alpha \colon U \longrightarrow B(C_n(X), V)$$

be a bounded $\pi_1(X)$-map. We define a family of linear maps by setting for each $e \in X$

$$\begin{aligned}\beta_e \colon W_e &\longrightarrow B(C_n(X)_e, V_e)\\ w &\longmapsto \Big((\sigma, x) \longmapsto \alpha_e(\gamma_{\sigma(0),x} \cdot \tau_{p(\sigma(0))}(\gamma_{x,\sigma(0)} \cdot w))(\sigma, x)\Big).\end{aligned}$$

Here $\gamma_{x,\sigma(0)}$ denotes the Element in $\pi_1(X)$ that corresponds to a path from x to $\sigma(0)$ in $\widetilde{X}$.

- By Remark 5.2.17, for each $e \in X$ and each $w \in W_e$, the map $\beta_e(w)$ is well-defined.
- For each $e \in X$ the map β_e is bounded and $\|\beta_e\|_\infty \le \|\alpha_e\|_\infty$, thus in particular $\|\beta\|_\infty \le \|\alpha\|_\infty$: We have for each $w \in W_e$ and each $g_0 \in \pi_1(X)$ with $t(g_0) = e$

$$\begin{aligned}\|\alpha_e(g_0 \cdot \tau_{s(g_0)}(g_0^{-1} \cdot w))\|_\infty &\le \|\alpha_e\|_\infty \cdot \|g_0 \cdot \tau_{s(g_0)}(g_0^{-1} \cdot w)\|_\infty\\ &\le \|\alpha_e\|_\infty \cdot \|\tau_{s(g_0)}\|_\infty \cdot \|(g_0^{-1} \cdot w)\|\\ &\le \|\alpha_e\|_\infty \cdot \|w\|.\end{aligned}$$

- The map β is $\pi_1(X)$-equivariant: We have for all $g \in \pi_1(X)$, $w \in W_{s(g)}$ and $(\sigma, x) \in Q_n(X)$ with $p(x) = t(g)$:

$$\begin{aligned}\beta_{t(g)}(g \cdot w)(\sigma, x) &= \alpha_{t(g)}(\gamma_{\sigma(0),x} \cdot \tau_{p(\sigma(0))}(\gamma_{x,\sigma(0)} \cdot g \cdot w))(\sigma, x)\\ &= g \cdot \alpha_{s(g)}(g^{-1} \cdot \gamma_{\sigma(0),x} \cdot \tau_{p(\sigma(0))}(\gamma_{x,\sigma(0)} \cdot g \cdot w))(\sigma, x)\\ &= g \cdot \alpha_{s(g)}(\gamma_{\sigma(0),g^{-1}\cdot x} \cdot \tau_{p(\sigma(0))}(\gamma_{g^{-1}\cdot x,\sigma(0)} \cdot w))(\sigma, x)\\ &= g \cdot \big(\alpha_{s(g)}(\gamma_{\sigma(0),g^{-1}\cdot x} \cdot \tau_{p(\sigma(0))}(\gamma_{g^{-1}\cdot x,\sigma(0)} \cdot w))(\sigma, g^{-1} \cdot x)\big)\\ &= g \cdot (\beta_{s(g)}(w)(\sigma, g^{-1} \cdot x))\\ &= (g \cdot \beta_{s(g)}(w))(\sigma, x)\end{aligned}$$

Hence $\beta_{t(g)}(g \cdot w) = g \cdot \beta_{s(g)}(w)$.

- For all $e \in X$, $w \in W_e$ and $(\sigma, x) \in Q_n(X)_e$ we have

$$\begin{aligned}(\beta_e \circ j_e)(w)(\sigma, x) &= \alpha_e(\gamma_{\sigma(0),x}\tau_{p(\sigma(0))}(\gamma_{x,\sigma(0)}j_e(w)))(\sigma, x)\\ &= \alpha_e(\gamma_{\sigma(0),x}\tau_{p(\sigma(0))}j_e(\gamma_{x,\sigma(0)}(w)))(\sigma, x)\\ &= \alpha_e(w)(\sigma, x).\end{aligned}$$

Hence $\alpha = \beta \circ j$. □

Corollary 5.2.19. Let X be a CW-complex and V a Banach $\pi_1(X)$-module. Then $((B(C_*(X), V')_{*\in\mathbb{N}}, \nu)$ is a strong, relatively injective $\pi_1(X)$-resolution of V'.

Proof. By Theorem 5.2.16, for each $x \in \widetilde{X}$, there is a strong, pointed cochain contraction

$$(s^*_x : B(C^{\text{sing}}_*(\widetilde{X}_x; \mathbb{R}), V'_{p(x)}) \longrightarrow B(C^{\text{sing}}_{*-1}(\widetilde{X}_x; \mathbb{R}), V'_{p(x)}))_{*\in\mathbb{N}}.$$

Hence, by Remark 5.2.11 and Proposition 5.2.15, $((B(C_*(X), V')_{*\in\mathbb{N}}, \nu)$ is a strong resolution of V'. By Proposition 5.2.18, it is also a relatively injective $\pi_1(X)$-resolution. □

Proposition 5.2.20. *Let X be a CW-complex. There is a canonical, norm non-increasing $\pi_1(X)$-chain map $\Phi_* : C_*(X) \longrightarrow C_*(\pi_1(X))$ extending the identity on $\mathbb{R}[X]$, given by*

$$\begin{aligned}\Phi^X_* : C_*(X) &\longrightarrow C_*(\pi_1(X))\\ (\sigma, x) &\longmapsto (\gamma_{\sigma(0),x}, \gamma_{\sigma(1),\sigma(0)}, \dots, \gamma_{\sigma(*),\sigma(*-1)}).\end{aligned}$$

Proof. The map Φ^X_* is well-defined and $\pi_1(X)$-equivariant by Remark 5.2.17. It maps simplices to simplices and is thus norm non-increasing. It is a chain map since for all $(\sigma, x) \in Q_n(X)$ and all $i \in \{0, \dots, n\}$ we have

$$\begin{aligned}\Phi^X_n(\partial_{i,n}\sigma, x) &= (\gamma_{\sigma(0),x}, \gamma_{\sigma(1),\sigma(0)}, \dots, \gamma_{\sigma(i),\sigma(i-1)} \cdot \gamma_{\sigma(i+1),\sigma(i)}, \dots, \gamma_{\sigma(*),\sigma(*-1)})\\ &= \partial_{i,n}\Phi^X_n(\sigma, x).\end{aligned}$$ □

Corollary 5.2.21 (Absolute Mapping Theorem for Groupoids)**.** Let X be a CW-complex and let V be a Banach $\pi_1(X)$-module. Then there is a canonical isometric isomorphism of graded semi-normed $\mathbb{R}$-modules

$$H^*_b(X; V') \longrightarrow H^*_b(\pi_1(X); V')$$

Proof. By Corollary 5.2.19 the cochain complex $B(C_*(X), V')$ is a strong, relatively injective resolution of V' and thus, by Theorem 3.4.10, there exists a norm non-increasing $\pi_1(X)$-cochain map $B(C_*(X), V') \longrightarrow B(C_*(\pi_1(X)), V')$ extending $\mathrm{id}_{V'}$. By Proposition 5.2.20, there exists a norm non-increasing $\pi_1(X)$-cochain map $B(C_*(\pi_1(X)), V') \longrightarrow B(C_*(X), V')$ extending $\mathrm{id}_{V'}$. By the fundamental lemma for groupoids, Proposition 3.4.7 these two maps induce canonical, mutually inverse, isometric isomorphisms in bounded cohomology. □

5.3 Relative Bounded Cohomology of Topological spaces

In this section, we prove the relative mapping theorem for certain pairs of CW-complexes, extending the result of Frigerio and Pagliantini [40, Proposition 4.4] to groupoids, and in particular, to the non-connected case. We will also consider general coefficients here instead of just coefficients in $\mathbb{R}$.

5.3.1 Bounded Cohomology of Pairs of Topological spaces

In this section, we will define bounded cohomology for π_1-injective pairs of CW-complexes:

Remark 5.3.1. Let $i\colon A \longrightarrow X$ be a CW-pair, i.e., let X be a CW-complex, A a CW-subcomplex and i the canonical inclusion. We call such a map π_1*-injective* if $\pi_1(i)$ is injective as a groupoid map (Definition 3.1.1). In our situation, this is equivalent to saying that for all $a \in A$, the map $\pi_1(i,a)\colon \pi_1(A,a) \longrightarrow \pi_1(X,a)$ between the fundamental groups at a is injective.

Let $i\colon A \longrightarrow X$ be a CW-pair and let i be π_1*-injective*. In particular, we can assume $\widetilde{A} \subset \widetilde{X}$. We consider now the groupoid pair $\pi_1(i)\colon \pi_1(A) \hookrightarrow \pi_1(X)$. Let V be a $\pi_1(X)$-module. Then

$$C^*(X,A;V) := (B(C_*(X),V), B(C_*(A),\pi_1(i)^*V), B(C_*(i),V))$$

is a $(\pi_1(X),\pi_1(A))$-cochain complex. We write $K^*(X,A;V)$ for the kernel of the map $C_b^*(i;V)\colon C_b^*(X;V) \longrightarrow C_b^*(A;\pi_1(i)^*V)$. Together with the induced norm, this defines a normed $\mathbb{R}$-cochain complex, Section 3.5.2.

We define a category $\mathsf{Top}^2\overline{\mathsf{Ban}}$ of CW-pairs (X,A) together with $\pi_1(X)$-modules similarly to $\mathsf{Top}\overline{\mathsf{Ban}}$.

Remark 5.3.2 (Functoriality)**.** Let $i\colon A \longrightarrow X$ and $j\colon B \longrightarrow Y$ be π_1-injective CW-pairs, $f\colon (X,A) \longrightarrow (Y,B)$ a continuous map and $\varphi\colon \pi_1(f)^*W \longrightarrow V$ a $\pi_1(X)$-map. Then the right-hand side of the following diagram commutes

$$\begin{tikzcd}
K^*(Y,B;W) \arrow[r] \arrow[d, dashed, "C_b^*(f;\varphi)|_{K^*(Y,B;W)}"] & C_b^*(Y;W) \arrow[r, "C_b^*(j;W)"] \arrow[d, "C_b^*(f;\varphi)"] & C_b^*(B;\pi_1(j)^*W) \arrow[d, "C_b^*(f|_A;\pi_1(i)^*\varphi)"] \\
K^*(X,A;V) \arrow[r] & C_b^*(X;V) \arrow[r, "C_b^*(i;V)"'] & C_b^*(A;\pi_1(i)^*V)
\end{tikzcd}$$

Hence, the cochain map on the left-hand side is defined. We write

$$K^*(f;\varphi) := C_b^*(f;\varphi)|_{K^*(Y,B;W)}.$$

This defines a contravariant functor $K^*\colon \mathsf{Top}^2\overline{\mathsf{Ban}} \longrightarrow {}_{\mathbb{R}}\mathsf{Ch}^{\|\cdot\|}$.

Definition 5.3.3. We call

$$H_b^*(X,A;V) := H^*(K^*(X,A;V))$$

endowed with the induced semi-norm, *the bounded cohomology of X relative to A with coefficients in V*. This defines a functor $\mathsf{Top}^2\overline{\mathsf{Ban}} \longrightarrow \mathbb{R}\text{-}\mathsf{Mod}_*^{\|\cdot\|}$.

Remark 5.3.4. Let $i\colon A \hookrightarrow X$ be a CW-pair. Gromov [45] defined bounded cohomology of X relative to A with coefficients in $\mathbb{R}$ as the cohomology of the kernel of the map

$$B(C_*^{\text{sing}}(i;\mathbb{R}),\mathbb{R})\colon B(C_*^{\text{sing}}(X;\mathbb{R}),\mathbb{R}) \longrightarrow B(C_*^{\text{sing}}(A;\mathbb{R}),\mathbb{R}),$$

endowed with the induced semi-norm. If the pair (X,A) is π_1-injective, our definition coincides with the one of Gromov by Proposition 5.2.6.

5.3.2 The Relative Mapping Theorem

Proposition 5.3.5. *Let $i\colon A \hookrightarrow X$ be a CW-pair and V a Banach $\pi_1(X)$-module. Then for each $n \in \mathbb{N}$, the Banach $(\pi_1(X),\pi_1(A))$-module $C^n(X,A;V)$ is relatively injective.*

Proof. Let the pair $(j,j')\colon (U,U',\varphi) \longrightarrow (W,W',\psi)$ be a relatively injective $(\pi_1(X),\pi_1(A))$-map and $(\tau,\tau')\colon (W,W',\psi) \longrightarrow (U,U',\varphi)$ a split for (j,j'). Furthermore, let $(\alpha,\alpha')\colon (U,U',\varphi) \longrightarrow C^n(X,A;V)$ be a $(\pi_1(X),\pi_1(A))$-map. As in the proof of Proposition 5.2.18, we define a $\pi_1(X)$-map by setting

$$\begin{aligned}\beta\colon W &\longrightarrow B(C_n(X),V)\\ w &\longmapsto \Big((\sigma,x) \longmapsto \alpha(\gamma_{\sigma(0),x}\cdot\tau(\gamma_{x,\sigma(0)}\cdot w))(\sigma,x)\Big).\end{aligned}$$

where $\gamma_{x,\sigma(0)}$ denotes the Element in $\pi_1(X)$ that corresponds to a path from x to $\sigma(0)$ in $\widetilde{X}$. Furthermore, we define a $\pi_1(A)$-map by setting

$$\begin{aligned}\beta'\colon W' &\longrightarrow B(C_n(A),V)\\ w &\longmapsto \Big((\sigma,x) \longmapsto \alpha'(\gamma'_{\sigma(0),x}\cdot\tau'(\gamma'_{x,\sigma(0)}\cdot w))(\sigma,x)\Big).\end{aligned}$$

Here $\gamma'_{x,\sigma(0)}$ denotes the element in $\pi_1(A)$ that corresponds to a path from x to $\sigma(0)$ in $\widetilde{A} \subset \widetilde{X}$. As in Proposition 5.2.18, $(\beta,\beta')\circ(j,j') = (\alpha,\alpha')$ and we have $\|\beta\|_\infty \leq \|\alpha\|_\infty$ and $\|\beta'\|_\infty \leq \|\alpha'\|_\infty$. For all $(\sigma,x) \in Q_n(A)$, the points $x,\sigma(0) \in \widetilde{A}$ are contained in the same connected component of $\widetilde{A} \subset \widetilde{X}$, therefore $\pi_1(i)(\gamma'_{\sigma(0),x}) = \gamma_{\sigma(0),x}$. Thus, for all $w \in W'$

$$\begin{aligned}\beta'(\psi(w)) &= \Big((\sigma,x) \longmapsto \alpha'(\gamma'_{\sigma(0),x}\cdot\tau'(\gamma'_{x,\sigma(0)}\cdot\psi(w)))(\sigma,x)\Big)\\ &= \Big((\sigma,x) \longmapsto \alpha'(\gamma'_{\sigma(0),x}\cdot\varphi\pi_1(i)^*\tau(\gamma_{x,\sigma(0)}\cdot w))(\sigma,x)\Big)\\ &= \Big((\sigma,x) \longmapsto B(C_*(i),V)\pi_1(i)^*\alpha(\gamma_{\sigma(0),x}\cdot\pi_1(i)^*\tau(\gamma_{x,\sigma(0)}\cdot w))(\sigma,x)\Big)\\ &= B(C_*(i),V)(\pi_1(i)^*\beta(w)).\end{aligned}$$

Hence (β,β') is a $(\pi_1(X),\pi_1(A))$-map. □

The following result is due to Frigerio and Pagliantini (for trivial coefficients), extending the construction of Ivanov:

Proposition 5.3.6 ([40])**.** *Let $i\colon A \hookrightarrow X$ be a pair of connected CW-complexes, such that i is π_1-injective and induces an isomorphism between the higher* *homotopy groups. Let V be a Banach module. Let $\widetilde{i}\colon \widetilde{A} \longrightarrow \widetilde{X}$ be the inclusion map. Then there exists a family of norm non-increasing, pointed cochain contractions*

$$\begin{pmatrix} \left(s_x^*\colon B(C_*^{\mathrm{sing}}(\widetilde{X};\mathbb{R}),V') \longrightarrow B(C_{*-1}^{\mathrm{sing}}(\widetilde{X};\mathbb{R}),V')\right)_{*\in\mathbb{N}_{>0}} \\ s_x^0\colon B(C_0^{\mathrm{sing}}(\widetilde{X};\mathbb{R}),V') \longrightarrow V' \end{pmatrix}_{x\in\widetilde{X}}$$

and a family of norm non-increasing, pointed cochain contractions

$$\begin{pmatrix} \left(\hat{s}_a^*\colon B(C_*^{\mathrm{sing}}(\widetilde{A};\mathbb{R}),V') \longrightarrow B(C_{*-1}^{\mathrm{sing}}(\widetilde{A};\mathbb{R}),V')\right)_{*\in\mathbb{N}_{>0}} \\ \hat{s}_a^0\colon B(C_0^{\mathrm{sing}}(\widetilde{A};\mathbb{R}),V') \longrightarrow V' \end{pmatrix}_{a\in\widetilde{A}}$$

that is compatible with the restriction to $\widetilde{A}$, i.e., the following diagram commutes for all $a \in \widetilde{A}$:

$$\begin{tikzcd} B(C_*^{\mathrm{sing}}(\widetilde{X}),V') \arrow[r, "s^*_{\widetilde{i}(a)}"] \arrow[d, "B(C_*^{\mathrm{sing}}(\widetilde{i}),V')"'] & B(C_{*-1}^{\mathrm{sing}}(\widetilde{X}),V') \arrow[d, "B(C_{*-1}^{\mathrm{sing}}(\widetilde{i}),V')"] \\ B(C_*^{\mathrm{sing}}(\widetilde{A}),V') \arrow[r, "\hat{s}_a^*"] & B(C_{*-1}^{\mathrm{sing}}(\widetilde{A}),V') \end{tikzcd}$$

Remark 5.3.7. The proof of Proposition 5.3.6 is a generalisation of Ivanov's proof of the fact that $B(C_*^{\mathrm{sing}}(\widetilde{X};\mathbb{R}),\mathbb{R})$ is a strong resolution of $\mathbb{R}$. We will discuss both results in Appendix A. Park [68, Lemma 4.2] stated without proof that, based on Ivanov's work, Proposition 5.3.6 also holds without the assumption on the higher homotopy groups. Pagliantini [66, Remark 2.29] demonstrates however, that Ivanov's proof cannot be generalised directly to the relative setting, see also Remark A.9.

Lemma 5.3.8. We can assume the family of cochain contractions $(s_x^*)_{x\in\widetilde{X}}$ and $(\hat{s}_a^*)_{a\in\widetilde{A}}$ in Proposition 5.3.6 to be pointed equivariant.

Proof. Since i is π_1-injective, we can identify $\mathrm{Deck}(A)$ with the subgroup of deck transformations in $\mathrm{Deck}(X)$ mapping $\widetilde{A}$ to itself. We proceed as in Remark 5.2.11: For each $x_0 \in X$, choose a lift $\widetilde{x_0} \in \widetilde{X}$, such that $\widetilde{x}_0 \in \widetilde{A}$ if $x_0 \in A$. Set for each $x_0 \in X$, $\beta \in \mathrm{Deck}(X)$ and each $\varphi \in B(C_*^{\mathrm{sing}}(\widetilde{X}),V')$

$$t^*_{\beta\cdot\widetilde{x}_0}(\varphi) = \beta * s^*_{\widetilde{x}_0}(\beta^{-1} * \varphi)$$

and similarly for each $a_0 \in A$, $\beta \in \mathrm{Deck}(A)$ and each $\varphi \in B(C_*^{\mathrm{sing}}(\widetilde{A}),V')$

$$\hat{t}^*_{\beta\cdot\widetilde{a}_0}(\varphi) = \beta * \hat{s}^*_{\widetilde{a}_0}(\beta^{-1} * \varphi)$$

Then $(t_x^*)_{x\in\widetilde{X}}$ and $(\hat{t}_a^*)_{a\in\widetilde{A}}$ are pointed equivariant families of cochain contractions. It is easy to see that $(t_x^*)_{x\in\widetilde{X}}$ and $(\hat{t}_a^*)_{a\in\widetilde{A}}$ are still compatible with the restriction to $\widetilde{A}$. ☐

Corollary 5.3.9. Let $i\colon A \hookrightarrow X$ be a CW-pair, such that i is π_1-injective and induces an isomorphism between the higher homotopy groups on each connected component of A. Let V be a Banach $\pi_1(X)$-module. Then $C^*(X,A;V')$ is a strong $(\pi_1(X),\pi_1(A))$-resolution of V'

Proof. Let $(s^*_x)_{x\in\widetilde{X}}$ and $(\hat{s}^*_a)_{a\in\widetilde{A}}$ be a pointed equivariant family of norm non-increasing cochain contractions as in Lemma 5.3.8. As in Proposition 5.2.15, we define norm non-increasing cochain contractions

$$\begin{aligned} \forall_{n\in\mathbb{N}_{>0}} \quad s^n\colon B(C_n(X),V') &\longrightarrow B(C_{n-1}(X),V') \\ \varphi &\longmapsto \Big((\sigma,x)\longmapsto s^n_x(\varphi\circ i_x)(\sigma)\Big) \\ s^0\colon B(C_0(X),V') &\longrightarrow V' \\ B(C_0(X),V')_e \ni \varphi &\longmapsto \varphi(\widetilde{e},\widetilde{e}) \end{aligned}$$

and

$$\begin{aligned} \forall_{n\in\mathbb{N}_{>0}} \quad \hat{s}^n\colon B(C_n(A),\pi_1(i)^*V') &\longrightarrow B(C_{n-1}(A),\pi_1(i)^*V') \\ \varphi &\longmapsto \Big((\sigma,x)\longmapsto \hat{s}^n_x(\varphi\circ i_x)(\sigma)\Big) \\ \hat{s}^0\colon B(C_0(A),\pi_1(i)^*V') &\longrightarrow \pi_1(i)^*V' \\ B(C_0(A),V')_e \ni \varphi &\longmapsto \varphi(\widetilde{e},\widetilde{e}). \end{aligned}$$

We only have to check that they commute with $B(C_*(i),V')$. For each $n\in\mathbb{N}$ and $\varphi\in B(C_n(X),V')$ we have

$$\begin{aligned} (\hat{s}^n\circ B(C_n(i),V'))(\varphi) &= \hat{s}^n(\varphi\circ C_n(i)) \\ &= \Big((\sigma,x)\longmapsto \hat{s}^n_x(\varphi\circ C_n(i)\circ i_x)(\sigma)\Big) \\ &= \Big((\sigma,x)\longmapsto (\hat{s}^n_x\circ B(C^{\text{sing}}_n(\widetilde{i}),V_{p(x)}))(\varphi\circ i_{\widetilde{i}(x)})(\sigma)\Big) \\ &= \Big((\sigma,x)\longmapsto (B(C^{\text{sing}}_{n-1}(\widetilde{i}),V_{p(x)})\circ s^n_{\widetilde{i}(x)})(\varphi\circ i_{\widetilde{i}(x)})(\sigma)\Big) \\ &= \Big((\sigma,x)\longmapsto s^n_{\widetilde{i}(x)}(\varphi\circ i_{\widetilde{i}(x)})(\widetilde{i}\circ\sigma)\Big) \\ &= \Big((\sigma,x)\longmapsto s^n(\varphi)(\widetilde{i}\circ\sigma,\widetilde{i}(x))\Big) \\ &= (B(C_{n-1}(i),V')\circ s^n)(\varphi). \end{aligned}$$ □

Lemma 5.3.10. Let $i\colon A\longrightarrow X$ be a π_1-injective CW-pair, and let V be a Banach $\pi_1(X)$-module. Then there exists a canonical, norm non-increasing $(\pi_1(X),\pi_1(A))$-cochain map

$$C^*(\pi_1(X),\pi_1(A);V)\longrightarrow C^*(X,A;V)$$

extending $(\mathrm{id}_V,\mathrm{id}_{\pi_1(i)^*V})$.

Proof. The map is given given by $(B(\Phi^X_*,V),B(\Phi^A_*,\pi_1(i)^*V))$, where the morphisms Φ^X_* and Φ^A_* are as in Proposition 5.2.20. Its easy to see that this is a $(\pi_1(X),\pi_1(A))$-map. □

Theorem 5.3.11 (Relative Mapping Theorem)**.** *Let $i\colon A\hookrightarrow X$ be a CW-pair, such that i is π_1-injective and induces isomorphisms between the higher homotopy groups on each connected component of A. Let V be a Banach $\pi_1(X)$-module. Then there is a canonical isometric isomorphism*

$$H^*_b(X,A;V')\longrightarrow H^*_b(\pi_1(X),\pi_1(A);V').$$

Proof. By Proposition 5.3.5 and Corollary 5.3.9, the $(\pi_1(X), \pi_1(A))$-cochain complex $C^*(X, A; V')$ is a strong, relatively injective $(\pi_1(X), \pi_1(A))$-resolution of V'. Therefore, by Proposition 3.5.24, there exists a norm non-increasing $(\pi_1(X), \pi_1(A))$-cochain map

$$\alpha^* \colon C^*(X, A; V') \longrightarrow C^*(\pi_1(X), \pi_1(A); V')$$

extending $(\mathrm{id}_{V'}, \mathrm{id}_{\pi_1(i)^*V'})$. By Lemma 5.3.10, there is a norm non-increasing $(\pi_1(X), \pi_1(A))$-cochain map $C^*(\pi_1(X), \pi_1(A); V') \longrightarrow C^*(X, A; V')$ extending $(\mathrm{id}_{V'}, \mathrm{id}_{\pi_1(i)^*V'})$. By the fundamental lemma for pairs, Proposition 3.5.24, these maps induce mutually inverse, norm non-increasing isomorphisms in bounded cohomology. In particular, the map $H_b^*(X, A; V') \longrightarrow H_b^*(\pi_1(X), \pi_1(A); V')$ induced by α^* is an isometric isomorphism. Also by the fundamental lemma, this isomorphism doesn't depend on the extension of $(\mathrm{id}_{V'}, \mathrm{id}_{\pi(i)^*V'})$. ☐

Corollary 5.3.12. Let $i \colon A \hookrightarrow X$ be a CW-pair, such that i is π_1-injective and induces isomorphisms between the higher homotopy groups on each connected component of A. Let V be a Banach $\pi_1(X)$-module. Let $\pi_1(A)$ be amenable. Then there is a canonical isometric isomorphism

$$H_b^*(X, A; V') \longrightarrow H_b^*(X; V').$$

Corollary 5.3.12 for was stated by Gromov [45] without the assumptions on i to be π_1-injective and to induce isomorphisms between the higher homotopy groups, but without a proof for the map to be isometric. This has been one motivation for us to study relative bounded cohomology in the groupoid setting in the first place. There is now, however, a short and beautiful proof of this stronger result by Bucher, Burger, Frigerio, Iozzi, Pagliantini and Pozzetti [18, Theorem 2]. This has also been shown independently by Kim and Kuessner [51, Theorem 1.2] via multicomplexes.

Proof. The following diagram commutes:

$$\begin{tikzcd}
H_b^n(X, A; V') \arrow[r] \arrow[d, "\cong"'] & H_b^n(X; V') \arrow[d, "\cong"] \\
H_b^n(\pi_1(X), \pi_1(A); V') \arrow[r, "\cong"'] & H_b^n(\pi_1(X); V')
\end{tikzcd}$$

Here, the column maps are the isometric isomorphisms induced by the (topological) mapping theorem, Theorem 5.3.11, and the row maps are induced by the canonical inclusions. The lower row map is an isometric isomorphisms by the algebraic mapping theorem, Corollary 4.2.5. ☐

Remark 5.3.13. One important reason to consider relative bounded cohomology is to study manifolds with boundary relative to the boundary, due to the relation between bounded cohomology and simplicial volume. We end this chapter by mentioning some examples of manifolds with boundary that satisfy the conditions of the relative mapping theorem:

(i) Compact aspherical 3-manifolds relative to a union of incompressible boundary components [3].

(ii) The relative hyperbolisation construction of Davis, Januszkiewicz and Weinberger [29] gives rise to many exotic examples. Let X be a manifold with boundary Y and assume that each connected component of Y is aspherical. Then the relative hyperbolisation $J(X, Y)$ relative to Y satisfies the assumption of Corollary 5.3.12.

(iii) Compact hyperbolic manifolds with totally geodesic boundary relative to the boundary [6, Proposition 13.1].

Chapter 6

Uniformly Finite Homology and Cohomology

Uniformly finite homology is a coarse homological invariant, introduced by Block and Weinberger [10] to study large-scale properties of metric spaces. It is a quasi-isometry invariant and can thus be defined also for finitely generated groups, considering a word metric on the group. As we will see, uniformly finite homology can be used to study geometric properties of groups.

One important property of uniformly finite homology is, that the zero degree uniformly finite homology group $H_0^{\mathrm{uf}}(X;\mathbb{R})$ of a metric space X vanishes, if and only if X is not amenable [10], confer Section 6.1. Other applications include rigidity properties of metric spaces of bounded geometry [36, 78] and the construction of aperiodic tilings for non-amenable spaces [10, 31].

Furthermore, Dranishnikov [33, 34, 35] studies the comparison map between homology and uniformly finite homology to derive results about the macroscopic dimension of manifolds.

In the first section of this chapter, we define uniformly finite homology and discuss some elementary properties and the theorem by Block and Weinberger about amenability and uniformly finite homology. Particularly, we show that uniformly finite homology for a finitely generated group is just homology with coefficients in ℓ^∞.

In the second section, we study the comparison map between homology and uniformly finite homology and calculate the uniformly finite homology of free groups and surface groups. We also show that for amenable groups the transfer map is a left inverse to the comparison map.

In the third section, we present bounded valued cohomology, introduced by Gersten, and analyse its relation both to bounded cohomology and uniformly finite homology. In particular, we show that, for finitely generated groups, it coincides with group cohomology with coefficients in ℓ^∞, therefore we see that in this setting bounded valued cohomology is dual to uniformly finite homology. We also discuss the results of Gersten and Mineyev about the relation between bounded valued cohomology (and bounded cohomology) and hyperbolic groups. In the Section 6.4, we recall the definition of quasi-morphisms and see that quasi-morphism embed into cohomology with ℓ^∞-coefficients, mirroring the result for bounded cohomology. In the fifth section, we discuss the relation between uniformly finite homology with real and with integer coefficients, following Whyte.

Sections 6.6 and 6.7 contain joint work with Francesca Diana. In the first part, we show that uniformly finite homology of amenable groups is often infinite dimensional. We also prove that, if M is a closed irreducible 3-manifold, either $\pi_1(M)$ is finite or $H_2^{\mathrm{uf}}(\pi_1(M);\mathbb{R})$ is infinite dimensional. In the second part, we study classes in zero degree uniformly finite homology of amenable groups. We distinguish classes that are "visible" or "invisible" to means and show that, if the group is infinite, there are infinitely many of both types of classes, giving an explicit construction in the later case.

6.1 Uniformly Finite Homology

In this section, we will introduce uniformly finite homology and discuss the theorem by Block and Weinberger [10], saying that zero degree uniformly finite homology characterises amenable UDBG spaces. We will also mention some fundamental properties of uniformly finite homology like the quasi-isometry invariance and the relation between uniformly finite homology and homology with ℓ^∞-coefficients.

In the following, if (X,d) is a metric space and $n \in \mathbb{N}$, we will always endow X^{n+1} with the maximum metric

$$\begin{aligned} d_{n+1}\colon X^{n+1} \times X^{n+1} &\longrightarrow \mathbb{R} \\ (x,y) &\longmapsto \max_{i\in\{0,\dots,n\}} d(x_i,y_i). \end{aligned}$$

We will consider uniformly finite homology with coefficients in normed Abelian groups:

Definition 6.1.1. A *normed Abelian group* is an Abelian group A together with a function $\|\cdot\|\colon A \longrightarrow \mathbb{R}_{\geq 0}$ such that

(i) For all $a \in A$, we have $\|a\| = 0$ if and only if $a = 0$.

(ii) For all $a \in A$, we have $\|-a\| = \|a\|$.

(iii) For all $a, b \in A$, we have $\|a+b\| \leq \|a\| + \|b\|$.

We call such a function $\|\cdot\|$ also a *norm on A*.

For us, the most important examples for normed Abelian groups will be normed $\mathbb{R}$-vector spaces and $\mathbb{Z}$ equipped with the norm induced by the Euclidean norm on $\mathbb{R}$.

Definition 6.1.2. Let A be a normed Abelian group and X a set. We write

$$\ell^\infty(X,A) := \bigl\{f\colon X \longrightarrow A \bigm| \sup_{x\in X} \|f(x)\| < \infty\bigr\}.$$

For each $\varphi \in \ell^\infty(X,A)$, write $\|\varphi\|_\infty := \sup_{x\in X} \|\varphi(x)\|$. Then $\ell^\infty(X,A)$ together with $\|\cdot\|_\infty$ is again a normed Abelian group. If A is a normed $\mathbb{R}$-module, so is $(\ell^\infty(X,A), \|\cdot\|_\infty)$. We also write $\ell^\infty(X) := \ell^\infty(X,\mathbb{R})$.

Definition 6.1.3. Let A be a normed Abelian group.

(i) Let (Y,d) be a metric space. We call a function $c \in \ell^\infty(Y,A)$ of *bounded geometry* if

$$\forall_{r\in\mathbb{R}_{>0}} \quad \sup_{y\in Y}\bigl|\{x \in B_r(y) \mid c(x) \neq 0\}\bigr| < \infty.$$

(ii) Let (X,d) be a metric space, $n \in \mathbb{N}$. We call a function $c \in \ell^\infty(X^{n+1},A)$ of *bounded diameter in X^{n+1}* if

$$\sup\bigl\{d(x_i,x_j) \mid x \in X^{n+1},\ c(x) \neq 0,\ i,j \in \{0,\dots,n\}\bigr\} < \infty.$$

Definition 6.1.4. Let (X,d) be a metric space and A a normed Abelian group.

(i) For each $n \in \mathbb{N}$ write $C_n^{\mathrm{uf}}(X;A) \subset \ell^\infty(X^{n+1},A)$ for the subspace of functions of bounded geometry and of bounded diameter in X^{n+1}. For $n \in \mathbb{Z}_{<0}$ set $C_n^{\mathrm{uf}}(X;A) = 0$.

(ii) We write a function $c \in C_n^{\mathrm{uf}}(X;A)$ also as a formal sum $\sum_{x\in X^{n+1}} c(x)\cdot x$.

(iii) Define a family of boundary operators

$$\partial_*\colon C_*^{\mathrm{uf}}(X;A) \longrightarrow C_{*-1}^{\mathrm{uf}}(X;A)$$

by setting for each $n \in \mathbb{N}_{>0}$ and for each $x \in X^{n+1}$

$$\partial_n(x) = \sum_{i=0}^{n}(-1)^i(x_0,\dots,\widehat{x}_i,\dots,x_n).$$

and extend to $C_n^{\mathrm{uf}}(X;A)$ in the obvious way (see Remark 6.1.5). For $n \in \mathbb{Z}_{<1}$ set $\partial_n = 0$. In this fashion, we get a chain complex.

(iv) The homology of $\bigl((C_n^{\mathrm{uf}}(X;A),\partial_n)_{n\in\mathbb{Z}}\bigr)$ is called *uniformly finite homology of X with coefficients in A* and denoted by $H_*^{\mathrm{uf}}(X;A)$.

Remark 6.1.5. The boundary maps are indeed well-defined: For $n \in \mathbb{N}_{>0}$ and $i \in \{1,\dots,n-1\}$ set

$$\begin{aligned}\partial_n^i\colon C_n^{\mathrm{uf}}(X;A) &\longrightarrow C_{n-1}^{\mathrm{uf}}(X;A)\\ \varphi &\longmapsto \Bigl(x \longmapsto \sum_{y\in X}\varphi(x_0,\dots,x_{i-1},y,x_i,\dots,x_{n-1})\Bigr),\end{aligned}$$

similarly for $i \in \{0,n\}$. Fix $\varphi \in C_n^{\mathrm{uf}}(G;\mathbb{R})$. Since φ is of bounded geometry and bounded diameter, there is a constant $C \in \mathbb{N}$, such that for each $x \in X^n$ there are at most C non-zero summands in $\sum_{y\in X}\varphi(x_0,\dots,x_{i-1},y,x_i,\dots,x_{n-1})$. In particular the sum is finite and $\|\partial_n^i(\varphi)\|_\infty \leq C\cdot\|\varphi\|_\infty$. Seeing that φ is bounded, so is $\partial_n^i(\varphi)$. Since φ is of bounded diameter, the same is true for $\partial_n^i(\varphi)$.

Because φ is of bounded diameter, we can define

$$D := \sup\bigl\{d(x_i,x_j) \mid x \in X^{n+1},\ \varphi(x) \neq 0,\ i,j \in \{0,\dots,n\}\bigr\} \in \mathbb{R}.$$

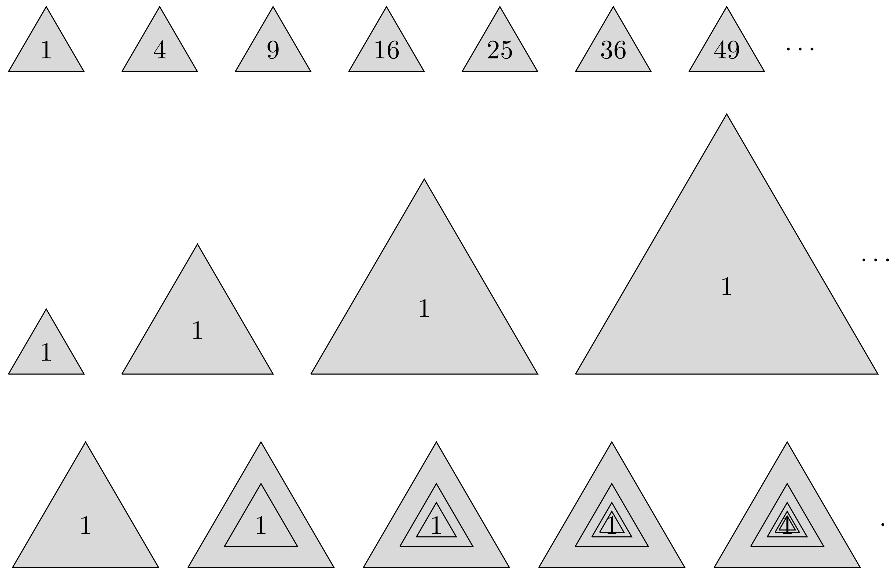

Figure 6.1: Functions that do *not* represent uniformly finite chains in $C_2^{\mathrm{uf}}(\mathbb{R}^2;\mathbb{R})$

For all $x \in X^n$ and $y \in X$ write $x_y := (x_0, \dots, x_{i-1}, y, x_i, \dots, x_{n-1}) \in X^{n+1}$. For all $r \in \mathbb{R}_{>0}$ and $z \in X^n$ we have

$$\{x \in B_r(z) \mid \partial_n^i \varphi(x) \neq 0\} \subset \{x \in B_r(z) \mid \exists_{y(x) \in X} \quad \varphi(x_{y(x)}) \neq 0\}. \qquad (*)$$

Furthermore, for $x \in B_r(z)$ and $y(x) \in X$ such that $\varphi(x_{y(x)}) \neq 0$ we have

$$\begin{aligned} d_{n+1}(x_{y(x)}, z_{z_0}) &\leq d_n(x, z) + d(y(x), z_0) \\ &\leq r + d(y(x), x_0) + d(x_0, z_0) \leq 2 \cdot r + D. \end{aligned}$$

Hence the following injection is well-defined:

$$\begin{aligned} \{x \in B_r(z) \mid \exists_{y(x) \in X} \quad \varphi(x_{y(x)}) \neq 0\} &\longrightarrow \{w \in B_{2 \cdot r + D}(z_{z_0}) \mid \varphi(w) \neq 0\} \\ x &\longmapsto x_{y(x)}. \end{aligned} \qquad (**)$$

By $(*)$ and $(**)$ we have

$$\big|\{x \in B_r(z) \mid \partial_n^i \varphi(x) \neq 0\}\big| \leq \big|\{w \in B_{2 \cdot r + D}(z_{z_0}) \mid \varphi(w) \neq 0\}\big|.$$

Therefore, since φ is of bounded geometry, also $\partial_i^n(\varphi)$ is of bounded geometry and ∂_n^i is thus a well-defined map for all $n \in \mathbb{N}_{>0}$ and $i \in \{0, \dots, n\}$. Set

$$\partial_n := \sum_{i=0}^{n} (-1)^n \cdot \partial_i^n.$$

One important property of uniformly finite homology is that it is a quasi-isometry invariant:

Proposition 6.1.6 ([10, Proposition 2.1]). *Let X and Y be metric spaces and A a normed Abelian group.*

(i) Let $f\colon X \longrightarrow Y$ be a quasi-isometric embedding. Then f induces a chain map

$$\begin{aligned} C^{\mathrm{uf}}_*(f;A)\colon C^{\mathrm{uf}}_n(X;A) &\longrightarrow C^{\mathrm{uf}}_n(Y;A) \\ c &\longmapsto \sum_{y\in Y}\Biggl(\sum_{x\in f^{-1}(y)} c(x)\Biggr)\cdot y. \end{aligned}$$

(ii) Let $f,g\colon X \longrightarrow Y$ be two quasi-isometric embeddings, at bounded distance from each other, that is, such that $\sup_{x\in X} d_Y(f(x),g(x)) < \infty$. Then $C^{\mathrm{uf}}_*(f;A)$ and $C^{\mathrm{uf}}_*(g;A)$ are chain homotopic.

(iii) In particular, if $f\colon X \longrightarrow Y$ is a quasi-isometry, the map induced by f in uniformly finite homology via $C^{\mathrm{uf}}_*(f;A)$

$$H^{\mathrm{uf}}_*(f;A)\colon H^{\mathrm{uf}}_*(X;A) \longrightarrow H^{\mathrm{uf}}_*(Y;A)$$

is an isomorphism of graded Abelian groups. If A is a normed vector space, $H^{\mathrm{uf}}_*(f;A)$ is an isomorphism of graded vector spaces.

Definition 6.1.7. Let (X,d) be a metric space.

(i) We call (X,d) *uniformly discrete* if

$$\exists_{r\in\mathbb{R}_{>0}} \quad \forall_{x\in X} \quad |B_r(x)| = 1.$$

(ii) We call (X,d) of *bounded geometry* if χ_X, and therefore every function in $\ell^\infty(X)$, is of bounded geometry.

(iii) If (X,d) is both uniformly discrete and of bounded geometry, we say that (X,d) is a *UDBG space*.

The main example of UDBG spaces for us are finitely generated groups with a word metric:

Definition 6.1.8 (Word metric). Let G be a group with a finite generating set S. The *word metric on G with respect to S* is the metric

$$\begin{aligned} d_S\colon G\times G &\longrightarrow \mathbb{R} \\ (g,h) &\longmapsto \inf\{n\in\mathbb{N} \mid \exists_{s_1,\dots,s_n\in S\cup S^{-1}} \quad g^{-1}\cdot h = s_1\cdots s_n\}. \end{aligned}$$

Remark 6.1.9. A finitely generated group with a word metric is obviously uniformly discrete and of bounded geometry. The word metric structure on a finitely generated group is unique up to quasi-isometry, hence the uniformly finite homology of a finitely generated group is (up to canonical isomorphism, induced by the identity map) uniquely defined.

Definition 6.1.10. Let X be a UDBG space. We call X *amenable* if there exists a sequence $(F_i)_{i\in\mathbb{N}}$ of non-empty finite subsets, such that for all $r\in\mathbb{R}_{>0}$

$$\lim_{i\to\infty}\frac{|\partial_r F_i|}{|F_i|} = 0.$$

Such a sequence $(F_i)_{i\in\mathbb{N}}$ is called a *Følner-sequence for X*.

If G is a finitely generated group with a word metric, then G is amenable as a groupoid, Definition 4.1.3, if and only if it is amenable as a UDBG space [26, Theorem 4.9.2], see also Section 6.7.2. Block and Weinberger have shown, that uniformly finite homology characterizes the amenability of a metric space:

Theorem 6.1.11 ([10, Theorem 3.1])**.** *Let (X, d) be a uniformly discrete space of bounded geometry. Then the following are equivalent:*

(i) The group $H_0^{\mathrm{uf}}(X; \mathbb{R})$ vanishes.

(ii) The fundamental class $[\chi_X] \in H_0^{\mathrm{uf}}(X; \mathbb{R})$ vanishes.

(iii) The space (X, d) is not amenable.

The following Proposition was noted by Brodzki, Niblo and Wright [12] for coefficients in $\mathbb{R}$, the proof is the same for any normed Abelian group, however:

Proposition 6.1.12 (Uniformly finite homology coincides with ℓ^∞-homology)**.** *Let G be a finitely generated group and A a normed Abelian group. There is a canonical chain isomorphism*

$$C_*^{\mathrm{uf}}(G; A) \longrightarrow C_*(G; \ell^\infty(G, A)).$$

In particular, $H_*^{\mathrm{uf}}(G; A) \cong H_*(G; \ell^\infty(G, A))$.

Here, the G-action on $\ell^\infty(G, A)$ is given

$$\begin{aligned} G \times \ell^\infty(G, A) &\longrightarrow \ell^\infty(G, A) \\ (g, \varphi) &\longmapsto \big(h \longmapsto \varphi(g^{-1} \cdot h)\big). \end{aligned}$$

The idea of the proof is, that for each $r \in \mathbb{N}_{>0}$ and $n \in \mathbb{N}$ there are only finitely many n-simplices having diameter smaller or equal r and 0-vertex e. Hence for each $c \in C_n^{\mathrm{uf}}(G; A)$ there are only finitely many G-orbits in G^{n+1} where c is not equal to 0 on the whole orbit. Thus, we can view c as a family of functions in $\ell^\infty(G, A)$ over a finite subset of G^{n+1} and this induces the chain isomorphism. See also Theorem 6.3.5 for the cohomological situation.

Corollary 6.1.13. Let G be a finitely generated group and A a normed Abelian group. Then uniformly finite homology in degree 0 is isomorphic to the space of coinvariants of $\ell^\infty(G, A)$, i.e.,

$$H_0^{\mathrm{uf}}(G; A) \cong \ell^\infty(G, A)_G.$$

With the description of uniformly finite homology in terms of ℓ^∞-homology one can give a short and direct proof of the vanishing theorem of Block and Weinberger in the case of finitely generated groups:

Theorem 6.1.14 (0-degree uniformly finite homology and amenable groups, [12])**.** *Let G be a finitely generated group. Then the following are equivalent:*

(i) The 0-th uniformly finite homology $H_0^{\mathrm{uf}}(G; \mathbb{R})$ vanishes.

(ii) The fundamental class $[\chi_G] \in H_0^{\mathrm{uf}}(G; \mathbb{R})$ vanishes.

(iii) The group G is not amenable.

Proof. Clearly (i) implies (ii). By Corollary 6.1.13, the dual space $H_0(G;\mathbb{R})'$ is isomorphic to $\operatorname{Hom}_G(\ell^\infty(G),\mathbb{R})$. An invariant mean on G is by definition an element $m \in \operatorname{Hom}_G(\ell^\infty(G),\mathbb{R})$ such that $m(\chi_G) = 1$, hence (ii) implies (iii). It is a simple fact that the set of invariant means spans $\operatorname{Hom}_G(\ell^\infty(G),\mathbb{R})$ [69, Proposition 2.2], so (iii) implies (i). □

We note the following fact since we will frequently use it in our examples:

Lemma 6.1.15. Let G be a finitely generated n-dimensional $\mathbb{R}$-Poincaré duality group. Then $H_k^{\mathrm{uf}}(G;\mathbb{R}) = 0$ for all $k \in \mathbb{N}_{>n}$ and $H_n^{\mathrm{uf}}(G;\mathbb{R}) \cong \mathbb{R}$.

Proof. This follows directly from our identification of uniformly finite homology with homology with coefficients in $\ell^\infty(G)$ and the fact that

$$H_n^{\mathrm{uf}}(G;\mathbb{R}) \cong H_n(G;\ell^\infty(G)) \cong H^0(G;\ell^\infty(G)) \cong \ell^\infty(G)^G \cong \mathbb{R}. \qquad \square$$

Motivated by Proposition 6.1.12, we will also use the following notation:

Definition 6.1.16. Let G be a group and A a normed Abelian group. We write $C^*_{\mathrm{uf}}(G;A) := C^*(G;\ell^\infty(G,A))$ and call $H^*_{\mathrm{uf}}(G;A) := H^*(C^*_{\mathrm{uf}}(G;A))$ *the uniformly finite cohomology of G with coefficients in A.*

6.2 Transfer and Comparison Maps

In this section, we discuss the comparison map between group homology and uniformly finite homology. If the group is amenable, each invariant mean induces a transfer map, which is a left inverse to the comparison map. This fact will be fundamental in Section 6.6. As an application, we demonstrate that the Hirsch rank of a nilpotent group is a quasi-isometry invariant.

We will show that for n-dimensional groups, the comparison map is injective in degree n and that for Poincaré n-duality groups, the same holds in degree $n-1$ as well. Using this we can calculate the uniformly finite homology of free groups and surface groups.

Proposition 6.2.1. *Let G be an amenable group. Then every left invariant mean $m \in M(G)$ induces a transfer map*

$$\begin{aligned} m_*\colon H_*(G;\ell^\infty(G)) &\longrightarrow H_*(G;\mathbb{R}) \\ [c\otimes\varphi] &\longmapsto m(\varphi)\cdot[c], \end{aligned}$$

which is left inverse to the comparison map

$$\begin{aligned} c_*^{\mathrm{uf}}\colon H_*(G;\mathbb{R}) &\longrightarrow H_*(G;\ell^\infty(G)) \\ [c] &\longmapsto [c\otimes\chi_G], \end{aligned}$$

induced by the canonical inclusion $\mathbb{R} \hookrightarrow \ell^\infty(G)$ as constant functions.

Proposition 6.2.1 follows directly from the following remark:

Remark 6.2.2. Let G be a group.

(i) By the short exact coefficient sequence of G-modules

$$0 \longrightarrow \mathbb{R} \longrightarrow \ell^\infty(G) \longrightarrow \ell^\infty(G)/\mathbb{R} \longrightarrow 0,$$

induced by the canonical inclusion $\mathbb{R} \hookrightarrow \ell^\infty(G)$ as constant functions, we get a long exact sequence in homology [15, Chapter III, Proposition 6.1]:

$$\cdots \longrightarrow H_n(G;\mathbb{R}) \longrightarrow H_n(G;\ell^\infty(G)) \longrightarrow H_n(G;\ell^\infty(G)/\mathbb{R}) \longrightarrow \cdots$$

(ii) We call the map

$$\begin{aligned} c_*^{\mathrm{uf}} \colon H_*(G;\mathbb{R}) &\longrightarrow H_*(G;\ell^\infty(G)) \\ [c] &\longmapsto [c \otimes \chi_G], \end{aligned}$$

induced by the canonical inclusion $\mathbb{R} \hookrightarrow \ell^\infty(G)$ as constant functions, *the comparison map in uniformly finite homology.* By a slight abuse of notation, we also write $c_*^{\mathrm{uf}} \colon H_*(G;\mathbb{R}) \longrightarrow H_*^{\mathrm{uf}}(G;\mathbb{R})$ for the map induced by the comparison map by Proposition 6.1.12.

(iii) Dually, the canonical inclusion $\mathbb{R} \hookrightarrow \ell^\infty(G)$ induces also a cohomological comparison map

$$c^*_{\mathrm{uf}} \colon H^*(G;\mathbb{R}) \longrightarrow H^*(G;\ell^\infty(G)).$$

(iv) Let G be amenable. Every left G-invariant mean $m \colon \ell^\infty(G) \longrightarrow \mathbb{R}$ splits the short exact coefficient sequence and hence we get for each $n \in \mathbb{N}$ a split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H_n(G;\mathbb{R}) \arrow[r] & H_n(G;\ell^\infty(G)) \arrow[r] \arrow[l, bend right, dashed, "H_n(G;m)"'] & H_n(G;\ell^\infty(G)/\mathbb{R}) \arrow[r] & 0. \end{tikzcd}$$

Corollary 6.2.3. Let G be a group of real cohomological dimension $n \in \mathbb{N}$.

(i) The comparison map c_n^{uf} is injective.

(ii) If G is a Poincaré duality group, the comparison map c_{n-1}^{uf} is injective as well.

Proof. This is follows directly from the long exact sequence of Remark 6.2.2, since $H_{n+1}(G;\ell^\infty(G)/\mathbb{R}) = 0$ and for a Poincaré duality group

$$H_n(G;\ell^\infty(G)/\mathbb{R}) \cong H^0(G;\ell^\infty(G)/\mathbb{R}) \cong (\ell^\infty(G)/\mathbb{R})^G.$$

But every G-invariant element in $\ell^\infty(G)$ is constant, hence $(\ell^\infty(G)/\mathbb{R})^G \cong 0$. $\square$

Using the identification of uniformly finite homology with ℓ^∞-homology we note:

Corollary 6.2.4. For any finitely generated amenable group G the comparison map $H_*(G;\mathbb{R}) \longrightarrow H_*^{\mathrm{uf}}(G;\mathbb{R})$ is injective.

Example 6.2.5.

(i) Let Σ be a hyperbolic surface group. Then

$$\dim_{\mathbb{R}} H_k^{\mathrm{uf}}(\Sigma;\mathbb{R}) = \begin{cases} 1 & \text{for } k=2 \\ \infty & \text{for } k=1 \\ 0 & \text{else.} \end{cases}$$

(ii) Let F be a non-Abelian free group. Then

$$\dim_{\mathbb{R}} H_k^{\mathrm{uf}}(F;\mathbb{R}) = \begin{cases} \infty & \text{for } k=1 \\ 0 & \text{else.} \end{cases}$$

Proof. Both Σ and F are non-amenable, hence $H_0^{\mathrm{uf}}(\Sigma;\mathbb{R}) = 0 = H_0^{\mathrm{uf}}(F;\mathbb{R})$ by Theorem 6.1.14. A surface group is of course a 2-dimensional Poincaré duality group hence $H_2^{\mathrm{uf}}(\Sigma;\mathbb{R}) \cong \mathbb{R}$ and $H_k^{\mathrm{uf}}(\Sigma;\mathbb{R}) = 0$ for $k \in \mathbb{N}_{>2}$ by Lemma 6.1.15. Furthermore, F is a 1-dimensional group, so the only interesting case is $k = 1$. The group Σ contains a family $(\Sigma_g)_{g\in\Gamma}$ of finite index subgroups that are surface groups of arbitrarily large genus and F contains a family $(F_{g'})_{g'\in\Gamma'}$ of finite index subgroups that are free of arbitrarily large rank. For all these groups, the comparison map in degree 1 is injective, hence we get inclusions for all $g \in \Gamma$ and $g' \in \Gamma'$

$$\begin{aligned} H_1(\Sigma_g;\mathbb{R}) &\hookrightarrow H_1^{\mathrm{uf}}(\Sigma_g;\mathbb{R}) \cong H_1^{\mathrm{uf}}(\Sigma;\mathbb{R}) \\ H_1(F_{g'};\mathbb{R}) &\hookrightarrow H_1^{\mathrm{uf}}(F_{g'};\mathbb{R}) \cong H_1^{\mathrm{uf}}(F;\mathbb{R}). \end{aligned}$$

The isomorphisms follow from the quasi-isometry invariance of uniformly finite homology. So $\dim_{\mathbb{R}} H_1^{\mathrm{uf}}(F;\mathbb{R}) = \infty = \dim_{\mathbb{R}} H_1^{\mathrm{uf}}(\Sigma;\mathbb{R})$. □

Definition 6.2.6. The *uniformly finite homological dimension of a group G* is defined as

$$\mathrm{hd}_{\mathrm{uf}}(G) = \sup\{n \in \mathbb{N} \mid H_n^{\mathrm{uf}}(G;\mathbb{R}) \neq 0\} \in \mathbb{N} \cup \{-\infty,+\infty\}.$$

Since uniformly finite homology is invariant under quasi-isometry, we obtain the following fact as a consequence of Proposition 6.2.1:

Corollary 6.2.7 ([9, Corollary 3.5])**.** The Hirsch rank of a finitely generated virtually nilpotent group G equals $\mathrm{hd}_{\mathrm{uf}}(G)$. In particular, the Hirsch rank is a quasi-isometry invariant.

Proof. We can assume that G is nilpotent, since by definition the Hirsch rank of a virtually nilpotent group is the Hirsch rank of any finite index nilpotent subgroup. Clearly, $\mathrm{hd}_{\mathrm{uf}}(G) \leq \mathrm{hd}_{\mathbb{R}}(G)$, where $\mathrm{hd}_{\mathbb{R}}(G)$ denotes the real homological dimension of G by Proposition 6.1.12.

On the other hand, for a finitely generated nilpotent group the real homological dimension is equal both to the largest $n \in \mathbb{N}$ for which $H_n(G;\mathbb{R}) \neq 0$ and to its Hirsch rank [73]. Since nilpotent groups are amenable, by Proposition 6.2.1, this integer must be smaller or equal than $\mathrm{hd}_{\mathrm{uf}}(G)$. □

6.3 ℓ^∞-Cohomology and Bounded Cohomology

Bounded valued cohomology was introduced by Gersten [42, 44] and seen by Gersten [43] and Mineyev [57] as a homological invariant that detects hyperbolicity of groups, i.e., it vanishes in certain cases if and only if the group is hyperbolic:

Theorem 6.3.1 ([57, 43])**.** *Let G be a finitely presented group. Then the following are equivalent:*

(i) The group G is hyperbolic.

(ii) For all normed $\mathbb{R}$-modules V we have $H^2_{(\infty)}(G;V) = 0$ and G.

(iii) The group G is of type $\mathcal{F}_\infty$ and for all normed $\mathbb{R}$-modules V and for all $n \in \mathbb{N}_{\geq 2}$ we have $H^n_{(\infty)}(G;V) = 0$

Mineyev [59] proved a different cohomological characterisation of hyperbolic groups in terms of the surjectivity of the comparison map in bounded cohomology and in the same article asked about the relation between these two invariants:

Theorem 6.3.2 ([58, 59])**.** *Let G be a finitely presented group. Then the following are equivalent*

(i) The group G is hyperbolic.

(ii) The comparison map $c^2_{b,V} \colon H^2_b(G;V) \longrightarrow H^2(G;V)$ is surjective for any normed G-module V.

(iii) The comparison maps $c^n_{b,V} \colon H^n_b(G;V) \longrightarrow H^n(G;V)$ are surjective for any $n \in \mathbb{N}_{\geq 2}$ and any normed G-module V.

Similarly to the situation for uniformly finite homology, we will show that bounded valued cohomology of a group equals cohomology with ℓ^∞-coefficients, hence bounded valued cohomology can be seen as a cohomological version of uniformly finite homology.

Using this, we will study the relation between bounded valued cohomology and bounded cohomology and show that Mineyev's result on the surjectivity of the comparison map in bounded cohomology immediately implies the vanishing of bounded valued cohomology. Furthermore, combining this with the result of Gersten, we directly see that surjectivity of the comparison maps implies hyperbolicity.

Bounded valued cohomology can be defined as a cellular version of bounded cohomology:

Definition 6.3.3. Let X be a CW-complex with finite n-skeleton for some $n \in \mathbb{N}$. Let V be a normed vector space.

(i) Consider the cellular chain complex $(C^{\mathrm{cell}}_*(\widetilde{X};V), \partial^{\mathrm{cell}}_*)_{*\in\mathbb{Z}}$, endowed in each degree with the ℓ^1-norm with respect to the cellular basis. Define for all $i \in \{0,\dots,n\}$
$$C^i_{(\infty)}(\widetilde{X};V) \subset C^i_{\mathrm{cell}}(\widetilde{X};V)$$
as the subspace of bounded cellular i-cochains, i.e., the subspace of all i-cochains that are bounded with respect to the ℓ^1-norm.

(ii) The coboundary maps of $C^*_{\mathrm{cell}}(\widetilde{X}, V)$ restrict to $(C^i_{(\infty)}(\widetilde{X}; V))_{i\in\{0,\dots,n\}}$, turning it into a cochain complex over $\{0,\dots,n\}$.

(iii) For $i \in \{0,\dots,n-1\}$, we call the corresponding cohomology $H^i_{(\infty)}(\widetilde{X}; V)$ the *bounded valued i-th cohomology group of $\widetilde{X}$ with coefficents in V.*

Definition 6.3.4. Let $n \in \mathbb{N}$ and G be a group. Assume that there is a CW-complex X that is a model for BG such that $X^{(n+1)}$ is finite. Then we set

$$H^n_{(\infty)}(G; V) := H^n_{(\infty)}(\widetilde{X}; V).$$

This is independent of the chosen model for BG by Corollary 6.3.6.

Similarly to the case of uniformly finite homology, bounded valued cohomology coincides with cohomology with ℓ^∞-coefficients:

Theorem 6.3.5. *Let $n \in \mathbb{N}$ and G be a group having a model of BG with finite n-skeleton. Let V be a normed G-module. Then there is a canonical isomorphism*

$$H^k(G; \ell^\infty(G, V)) \longrightarrow H^k_{(\infty)}(G; V)$$

for all $k \in \{0,\dots,n-1\}$.

Proof. Choose a CW-complex X that is a model for BG such that $X^{(n)}$ is finite. Up to canonical isomorphism, we can describe $H^*(G; \ell^\infty(G, V))$ as the cohomology of $C^*_{\mathrm{cell}}(X; \ell^\infty(G, V))$ (with twisted coefficients). There is a (canonical) pair of mutually inverse cochain isomorphisms

$$\begin{aligned}
\alpha^* \colon C^*_{\mathrm{cell}}(X; \ell^\infty(G, V)) &\longrightarrow C^*_{(\infty)}(\widetilde{X}; V)\\
\varphi &\longmapsto \big(\sigma \longmapsto \varphi(\sigma)(e)\big)\\
\beta^* \colon C^*_{(\infty)}(\widetilde{X}; V) &\longrightarrow C^*_{\mathrm{cell}}(X; \ell^\infty(G, V))\\
\psi &\longmapsto \Big(\sigma \longmapsto \big(g \longmapsto \psi(g^{-1}\cdot\sigma)\big)\Big).
\end{aligned}$$

- Consider $\varphi \in C^k_{\mathrm{cell}}(X; \ell^\infty(G, V))$ and $k \in \{0,\dots,n\}$. Then $\alpha^k(\varphi)$ is a bounded valued cochain, because: Let $F^{(k)}$ be a (finite) representative system for the action of G on the k-cells of $\widetilde{X}$. Then for all k-cells σ there is a $g \in G$ such that $g\cdot\sigma \in F^{(k)}$; hence

$$\begin{aligned}
|\alpha^k(\varphi)(\sigma)| &= |\varphi(\sigma)(e)|\\
&= |\varphi(g\cdot\sigma)(g)|\\
&\le \max_{\tau\in F^{(k)}} \|\varphi(\tau)\|_\infty
\end{aligned}$$

  Therefore, we have

$$\sup_{\sigma\in S_k(\widetilde{X})} |\alpha^k(\varphi)(\sigma)| \le \max_{\tau\in F^{(k)}} \|\varphi(\tau)\|_\infty < \infty.$$

- Consider $\psi \in C^k_{(\infty)}(\widetilde{X}; V)$. Then for all k-cells σ the map $\beta^k(\psi)(\sigma)$ is bounded, since for all $g \in G$

$$|\beta^k(\psi)(\sigma)(g)| = |\psi(g^{-1}\cdot\sigma)| \le \|\psi\|_\infty < \infty.$$

- The map $\beta^k(\psi)$ is G-equivariant, since for all k cells σ and all $h, g \in G$

$$\begin{aligned}\beta^k(\psi)(h\cdot\sigma)(g) &= \psi(g^{-1}\cdot h\cdot\sigma)\\ &= \beta^k(\psi)(\sigma)(h^{-1}\cdot g)\\ &= h\cdot\beta^k(\psi)(g).\end{aligned}$$

- The maps α^* and β^* are cochain maps: For all $k \in \{0,\dots,n\}$, for all $\sigma \in C^{\mathrm{cell}}_{k+1}(\widetilde{X};\mathbb{R})$ and all $\varphi \in C^k_{\mathrm{cell}}(X;V)$

$$\begin{aligned}\delta^k(\alpha_k(\varphi))(\sigma) &= \alpha_k(\varphi)(\partial_{k+1}\sigma)\\ &= \varphi(\partial_{k+1}\sigma)(e)\\ &= (\delta_k\varphi)(\sigma)(e)\\ &= \alpha_{k+1}(\delta_k\varphi)(\sigma).\end{aligned}$$

  Similarly for β^*.

- For all $k \in \{0,\dots,n\}$, the maps α_k and β_k are obviously mutually inverse: For all $\sigma \in C^{\mathrm{cell}}_k(\widetilde{X};\mathbb{R})$, all $g \in G$, all $\varphi \in C^k_{\mathrm{cell}}(X;\ell^\infty(G,V))$ and all $\psi \in C^k_{(\infty)}(\widetilde{X};V)$ we have

$$(\alpha_k\circ\beta_k)(\psi)(\sigma) = \beta_k(\psi)(\sigma)(e) = \psi(\sigma)$$

  and

$$(\beta_k\circ\alpha_k)(\varphi)(\sigma)(g) = \alpha_k(\varphi)(g^{-1}\cdot\sigma) = \varphi(g^{-1}\cdot\sigma)(e) = \varphi(\sigma)(g). \qquad \square$$

In particular, the definition of bounded valued cohomology is independent up to canonical isomorphism of the choice of the classifying space and the CW-structure on BG:

Corollary 6.3.6. Let $n \in \mathbb{N}$ and G be a group. Suppose X and Y are two models of BG with finite n-skeleton. Then there is a canonical isomorphism

$$H^k_{(\infty)}(\widetilde{X};V) \cong H^k_{(\infty)}(\widetilde{Y};V)$$

for all $k \in \{0,\dots,n-1\}$.

Corollary 6.3.7. Let G be a hyperbolic group and V a normed $\mathbb{R}$-module. Then G has a model X for BG such that $X^{(k)}$ is finite for all $k \in \mathbb{N}$. Let V be a normed $\mathbb{R}$-module. For all $n \in \mathbb{N}_{\geq 2}$ we have

$$H^n_{(\infty)}(G;V) \cong H^n(G;\ell^\infty(G,V)) = 0.$$

Proof. The existence of such a model X is a well-known result [11, Chapter III.Γ, Corollary 3.26]. By Theorem 6.3.2, the comparison map

$$c^n_{b,\ell^\infty(G,V)}\colon H^n_b(G;\ell^\infty(G,V)) \longrightarrow H^n(G;\ell^\infty(G,V))$$

is surjective. Since $\ell^\infty(G,V)$ is relatively injective G-module, $H^n_b(G;\ell^\infty(G,V))$ is trivial [62]. $\square$

The implication $(ii) \Longrightarrow (i)$ from Theorem 6.3.2 follows now directly from Gersten's part of Theorem 6.3.1:

Corollary 6.3.8. Let G be a finitely presented group. If the comparison map $c^2_{b,V}\colon H^2_b(G;V) \longrightarrow H^2(G;V)$ is surjective for any normed G-module V, then G is hyperbolic.

Proof. As in the proof of Corollary 6.3.7, we see that $H^n_{(\infty)}(G;V) = 0$ for all $n \in \mathbb{N}_{\geq 2}$, hence by Theorem 6.3.1, G is hyperbolic. $\square$

We have seen that the comparison map in bounded cohomology is surjective if and only if ℓ^∞-cohomology vanishes (if considering *all* coefficients). Moreover, as we have noted in Section 6.2, the comparison map in ℓ^∞-cohomology is injective for amenable groups and for these groups bounded cohomology vanishes. There is also the following relation between the comparison maps, noted by Gersten for bounded valued cohomology, which in our language is now obvious:

Proposition 6.3.9. *For $n \in \mathbb{N}_{>1}$ and all normed G-spaces V, the composition $c^n_{\mathrm{uf},V} \circ c^n_{b,V}$ of the comparison maps is zero.*

Proof. The following diagram commutes:

$$\begin{tikzcd}
H^n_b(G;V) \arrow[r] \arrow[d, "c^n_{b,V}"'] & H^n_b(G,\ell^\infty(G;V)) = 0 \arrow[d, "c^n_{b,\ell^\infty(G,V)}"] \\
H^n(G;V) \arrow[r, "c^n_{\mathrm{uf},V}"'] & H^n(G;\ell^\infty(G,V))
\end{tikzcd}$$

Here, we use again that $H^n_b(G;\ell^\infty(G,V))$ is trivial since $\ell^\infty(G,V)$ is relatively injective, [62]. $\square$

The examples mentioned above and Section 6.4 lead to the following questions:

Question 6.3.10. Let G be a finitely generated group and $n \in \mathbb{N}$. Under which conditions on G is the sequence

$$H^n_b(G;\mathbb{R}) \xrightarrow{c^n_{b,\mathbb{R}}} H^n(G;V) \xrightarrow{c^n_{\mathrm{uf},\mathbb{R}}} H^n(G;\ell^\infty(G,V))$$

exact? Is there a long exact sequence that relates the comparison maps in bounded cohomology and bounded valued cohomology?

6.4 Quasi-Morphisms and ℓ^∞-Cohomology

In this section we study the relation between quasi-morphisms and cohomology with coefficients in ℓ^∞. It turns out that this relation mirrors the situation in bounded cohomology.

Definition 6.4.1. Let G be a group. A *quasi-morphism from G to $\mathbb{R}$* is a map $\varphi\colon G \longrightarrow \mathbb{R}$ such that

$$D(\varphi) := \sup_{a,b\in G} |\varphi(a\cdot b) - \varphi(a) - \varphi(b)| < \infty.$$

We call $D(\varphi)$ the *defect of* φ. We call φ *homogeneous* if for all $n \in \mathbb{N}$ and $g \in G$

$$\varphi(g^n) = n \cdot \varphi(g).$$

and write $Q(G) \subset \mathrm{Map}(G,\mathbb{R})$ for the $\mathbb{R}$-subspace of homogeneous quasi-morphisms.

Our result will be motivated by the following fact:

Theorem 6.4.2 ([25, Theorem 2.50])**.** *For every group G there is an exact sequence*

$$0 \longmapsto H^1(G;\mathbb{R}) \longrightarrow Q(G) \longrightarrow H^2_b(G;\mathbb{R}) \longrightarrow H^2(G;\mathbb{R}).$$

The first map is given by the canonical inclusion

$$\begin{aligned} i\colon H^1(G;\mathbb{R}) \cong \mathrm{Hom}(G;\mathbb{R}) &\lhook\joinrel\longrightarrow Q(G) \\ \varphi &\longmapsto \big(g_1 \longmapsto \varphi(1, g_1^{-1})\big). \end{aligned}$$

Proof. This is a well-known result that follows directly from the definition of quasi-morphisms. □

Now we can show that $H^1_{\mathrm{uf}}(G;\mathbb{R})$ detects quasi-morphisms from G to $\mathbb{R}$. This result and its proof should be compared to the situation in bounded cohomology as expressed by Theorem 6.4.2.

Proposition 6.4.3. *For every group G, there is a canonical injection*

$$c\colon Q(G) \lhook\joinrel\longrightarrow H^1_{\mathrm{uf}}(G;\mathbb{R}),$$

such that $c \circ i = c^1_{\mathrm{uf}}$.

Proof. For all $\varphi\colon G \longrightarrow \mathbb{R}$ set

$$\begin{aligned} \hat{\varphi}\colon G &\longrightarrow \mathrm{Map}(G,\mathbb{R}) \\ g &\longmapsto \big(h \longmapsto \varphi(g^{-1}\cdot h)\big). \end{aligned}$$

We then define the map

$$\begin{aligned} c'\colon Q(G) &\longrightarrow C^1_{\mathrm{uf}}(G;\mathbb{R}) \\ \varphi &\longmapsto \big((g_0,g_1) \longmapsto \hat{\varphi}(g_0\cdot g_1) - \hat{\varphi}(g_0)\big) = \hat{\varphi}\circ\partial_1. \end{aligned}$$

Now $c'(\varphi)$ is G-equivariant since for all $g, g_0, g_1, h \in G$

$$c'(\varphi)(g\cdot g_0, g_1)(h) = \varphi(g_1^{-1}g_0^{-1}(g^{-1}h)) - \varphi(g_0^{-1}(g^{-1}h)) = c'(\varphi)(g_0,g_1)(g^{-1}h).$$

Furthermore, we have for all $g_0, g_1 \in G$

$$\begin{aligned}
&\sup_{h\in G} |\hat{\varphi}(g_0 \cdot g_1)(h) - \hat{\varphi}(g_0)(h)| \\
&\qquad = \sup_{h\in G} |\varphi((g_1^{-1} \cdot g_0^{-1} \cdot h) - \varphi(g_0^{-1} \cdot h)| \\
&\qquad \le \sup_{h\in G} |\varphi(g_1^{-1} \cdot g_0^{-1} \cdot h) - \varphi(g_0^{-1} \cdot h) - \varphi(g_1^{-1})| + |\varphi(g_1^{-1})| \\
&\qquad \le D(\varphi) + |\varphi(g_1^{-1})|.
\end{aligned}$$

Hence $c'(\varphi)(g_0, g_1) \in \ell^\infty(G)$. Obviously, for all $\varphi \in Q(G)$ the image $c'(\varphi)$ is a cocycle. Let $c\colon Q(G) \longrightarrow H^1_{\mathrm{uf}}(G;\mathbb{R})$ be the map induced by c'. By a similar calculation, we see that for $\varphi \in \mathrm{Hom}(G,\mathbb{R})$, i.e., $D(\varphi) = 0$, we have $c(i(\varphi)) = [(g_0, g_1) \longmapsto (h \longmapsto \varphi(g_1))] = c^1_{\mathrm{uf}}(\varphi)$, hence $c^1_{\mathrm{uf}} = c \circ i$. We finally show that c is injective: Consider any pair $\varphi \in Q(G)$ and $\psi \in C^0_{\mathrm{uf}}(G)$ such that $c'(\varphi) = \delta^1_{\mathrm{uf}}\psi$. Then

$$\begin{aligned}
\sup_{g_1\in G} |\varphi(g_1)| &= \sup_{g_1\in G} |\varphi(g_1) + \varphi(e) - \varphi(g_1 \cdot 1) - \varphi(e) + \varphi(g_1 \cdot 1)| \\
&\le \sup_{g_0,g_1\in G} \Big(|\varphi(g_1) + \varphi(g_0) - \varphi(g_0 \cdot g_1)| + |\delta^1_{\mathrm{uf}}\psi(1, g_1^{-1})(e)| \Big) \\
&= D(\varphi) + \sup_{g_1\in G} |\psi(g_1^{-1})(e) - \psi(1)(e)| \\
&\le D(\varphi) + 2\|\psi\|_\infty.
\end{aligned}$$

Thus $\varphi\colon G \longrightarrow \mathbb{R}$ is bounded and homogeneous, hence trivial. □

Corollary 6.4.4. Let G be a hyperbolic $\mathbb{R}$-Poincaré duality group of dimension $n \in \mathbb{N}_{>0}$. Then for $k \in \mathbb{N}$

$$\dim_{\mathbb{R}} H^{\mathrm{uf}}_k(G;\mathbb{R}) = \begin{cases} 1 & \text{for } k = n \\ \infty & \text{for } k = n-1 \\ 0 & \text{else.} \end{cases}$$

Proof. The vanishing for $k \in \{0, \dots, n-1\}$ is a consequence of duality and Corollary 6.3.7. Consider now $k = n - 1$. If G is quasi-isometric to $\mathbb{Z}$, this follows from our calculation for amenable groups (Proposition 6.7.2). If G is not elementary hyperbolic, then $Q(G)$ is infinite-dimensional by the result of Epstein and Fujiwara [37], so by Proposition 6.4.3 it follows that $\dim_{\mathbb{R}} H^1_{\mathrm{uf}}(G;\mathbb{R}) = \infty$. □

6.5 Integral and Real Coefficients

For some applications (e.g. Dranishnikov's analysis of large-scale dimensions [34] or the quasi-isometry classification of certain group extensions by Kleiner and Leeb [52]) it is important to consider uniformly finite homology with coefficients in $\mathbb{Z}$. The description in terms of ℓ^∞-homology allows us to present concisely what is known about the relation between coefficients in $\mathbb{Z}$ and in $\mathbb{R}$. Furthermore, we discuss an important geometric characterisation for a class in degree 0 to be trivial. We basically follow Whyte [78] in this section.

Proposition 6.5.1. *Let G be a finitely generated group.*

(i) Let

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{p} C \longrightarrow 0$$

be a short exact sequence of normed abelian groups and $\sigma\colon C \longrightarrow B$ a bounded, not necessarily linear, splitting. Then there is an induced long exact sequence:

$$\cdots \longrightarrow H_n^{\mathrm{uf}}(G;A) \longrightarrow H_n^{\mathrm{uf}}(G;B) \longrightarrow H_n^{\mathrm{uf}}(G;C) \longrightarrow H_{n-1}^{\mathrm{uf}}(G;A) \longrightarrow \cdots$$

(ii) In particular, there is a long exact sequence:

$$\cdots \longrightarrow H_n^{\mathrm{uf}}(G;\mathbb{Z}) \longrightarrow H_n^{\mathrm{uf}}(G;\mathbb{R}) \longrightarrow H_n^{\mathrm{uf}}(G;\mathbb{R}/\mathbb{Z}) \longrightarrow H_{n-1}^{\mathrm{uf}}(G;\mathbb{Z}) \longrightarrow \cdots$$

Proof. Applying ℓ^∞ to the coefficient sequence induces a short exact sequence of normed G-modules:

$$0 \longrightarrow \ell^\infty(G,A) \longrightarrow \ell^\infty(G,B) \longrightarrow \ell^\infty(G,C) \longrightarrow 0.$$

The only non-obvious part is the surjectivity of $\ell^\infty(G,B) \longrightarrow \ell^\infty(G,C)$ and this follows from the existence of the splitting. Thus we get a long exact sequence in homology.

A splitting for the second part is given by mapping each class in $\mathbb{R}/\mathbb{Z}$ to its representative in $[0,1)$. □

Corollary 6.5.2. Let G be an infinite finitely generated group. Then the change-of-coefficients map $H_0^{\mathrm{uf}}(G;\mathbb{Z}) \longrightarrow H_0^{\mathrm{uf}}(G;\mathbb{R})$ is surjective.

Proof. We only have to show that $H_0^{\mathrm{uf}}(G,\mathbb{R}/\mathbb{Z}) \cong 0$. But uniformly finite homology of an infinite group with bounded coefficients vanishes in degree 0, since: In an infinite group, one can always find a locally finite family $(t_g)_{g\in G}$ of proper edge path rays ("tails"), such that $\partial_1 t_g = g$ for all $g \in G$. Since the the coefficient module $\mathbb{R}/\mathbb{Z}$ is bounded, the formal sum $\sum_{g\in G} t_g$ is a uniformly finite 1-chain, whose boundary is the fundamental class $[\chi_G]$. This is similar to the argument in [10, Lemma 2.4]. □

Definition 6.5.3. Let G be a finitely generated discrete group with a word metric d. For any $S \subset G$ and $r \in \mathbb{R}_{>0}$ set

$$\partial_r(S) := \{g \in G \mid 0 < d(g,S) \leq r\}.$$

The following Theorem of Whyte gives a geometric characterisation of the vanishing of zero cycles in uniformly finite homology:

Theorem 6.5.4 ([78, Theorem 7.7])**.** *Let G be a finitely generated group. A cycle $c \in C_0^{\mathrm{uf}}(G;\mathbb{Z})$ is trivial in $H_0^{\mathrm{uf}}(G;\mathbb{Z})$ if and only if*

$$\exists_{C,\, r\in\mathbb{N}_{>0}} \quad \forall_{S\subseteq G \text{ finite}} \quad \left|\sum_{s\in S} c(s)\right| \leq C \cdot |\partial_r S|. \tag{W}$$

We will give a short proof of the fact that condition (W) in Theorem 6.5.4 is necessary for a cycle to vanish that works both for and coefficients in $\mathbb{Z}$ and in $\mathbb{R}$:

Proof that vanishing of a cycle implies condition (W). Let $c \in C_0^{\mathrm{uf}}(G;\mathbb{Z})$ be a boundary. Consider a chain $b \in C_1^{\mathrm{uf}}(G;\mathbb{Z})$ such that $\partial_1^{\mathrm{uf}} b = c$. Choose $r \in \mathbb{R}_{>0}$ such that $b(g_0, g_1) = 0$ for all $g_0, g_1 \in G$ with $d(g_0, g_1) \geq r$. Set $C := \|c\|_\infty$. If $S \subset G$ is a finite subset in G, the points in $\partial_r S$ are the only points in $G \setminus S$ that can be reached by edges in the support of b starting in S, i.e., if $b(g_0, g_1) \neq 0$ and $g_0 \in S$, then $g_1 \in \partial_r S \cup S$, Figure 6.5, hence

$$\left|\sum_{s \in S} c(s)\right| \leq \sum_{s \in \partial_r S} |c(s)| \leq C \cdot |\partial_r S|.$$

The proof is the same for $\mathbb{R}$-coefficients. □

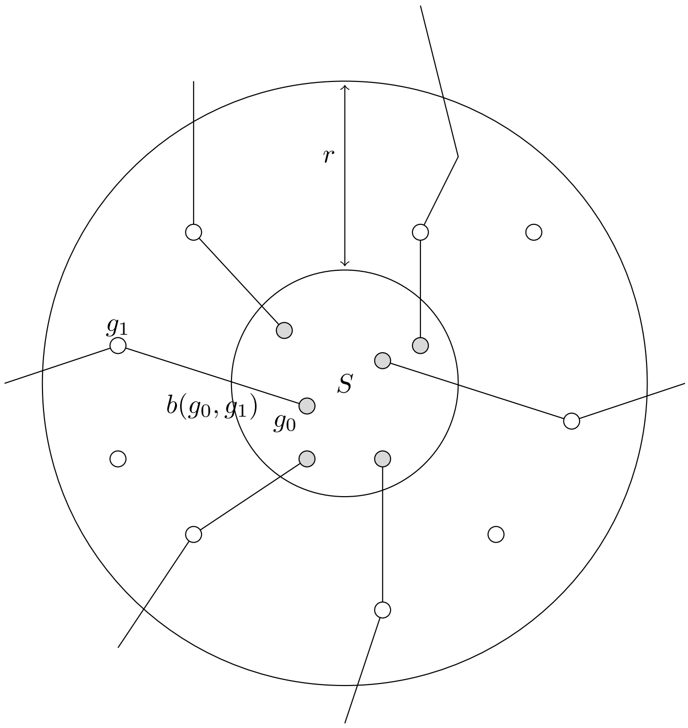

Figure 6.2: Whyte's vanishing criterion

Whyte stated Theorem 6.5.4 only for coefficients in $\mathbb{Z}$, but it is now clear that it also holds for coefficients in $\mathbb{R}$:

Corollary 6.5.5. Theorem 6.5.4 also holds if one replaces $\mathbb{Z}$ by $\mathbb{R}$.

Proof. Let $c \in C_0^{\mathrm{uf}}(G;\mathbb{R})$ be a cycle. We have already seen that condition (W) is necessary for the class $[c]$ to be trivial in $H_0^{\mathrm{uf}}(G;\mathbb{R})$. Assume that c satisfies (W) for some $C, r \in \mathbb{R}_{>0}$. Let $i\colon C_0(G;\mathbb{Z}) \longrightarrow C_0(G;\mathbb{R})$ denote the change-of-coefficients map. By Corollary 6.5.2, there is a cycle $c_{\text{int}} \in C_0^{\text{uf}}(G;\mathbb{Z})$ such that $[c - i(c_{\text{int}})] = 0$ in $H_0^{\text{uf}}(G;\mathbb{R})$. Hence $c - i(c_{\text{int}})$ also satisfies (W) for some constants $C', r' \in \mathbb{R}_{>0}$. Thus for all finite subsets $S \Subset G$ and $r'' := \max\{r, r'\}$ and $C'' := C + C'$

$$\begin{aligned}\left|\sum_{s\in S} c_{\text{int}}(s)\right| &\leq \left|\sum_{s\in S} c(s)\right| + \left|\sum_{s\in S}(c - i(c_{\text{int}}))(s)\right| \\ &\leq C \cdot |\partial_r S| + C' \cdot |\partial_{r'} S| \\ &\leq C'' \cdot |\partial_{r''} S|.\end{aligned}$$

Hence $[c_{\text{int}}]$ is trivial in $H_0^{\text{uf}}(G;\mathbb{Z})$ by Theorem 6.5.4 and therefore also $[c] = 0$ in $H_0^{\text{uf}}(G;\mathbb{R})$. □

Corollary 6.5.6 ([78, Lemma 7.7])**.** Let G be a finitely generated group. Then the change-of-coefficients map $H_0^{\text{uf}}(G;\mathbb{Z}) \longrightarrow H_0^{\text{uf}}(G;\mathbb{R})$ is an isomorphism.

Proof. It is surjective by Corollary 6.5.2 and injective by Theorem 6.5.4 and Corollary 6.5.5. □

The following corollary was noted by Gersten for bounded valued cohomology:

Corollary 6.5.7. Let G be a finitely generated Poincaré duality group of dimension $n \in \mathbb{N}$. Then for all $k \in \{0, \dots, n-1\}$ the change-of-coefficients map

$$H_k^{\text{uf}}(G;\mathbb{Z}) \longrightarrow H_k^{\text{uf}}(G;\mathbb{R})$$

is an isomorphism.

Proof. The module $\mathbb{R}/\mathbb{Z}$ is bounded, hence

$$\ell^\infty(G, \mathbb{R}/\mathbb{Z}) = \operatorname{Map}(G, \mathbb{R}/\mathbb{Z}) \cong_{\mathbb{Z}G} \operatorname{Hom}(\mathbb{Z}G, \mathbb{R}/\mathbb{Z}).$$

So for all $k \in \{0, \dots, n-1\}$

$$\begin{aligned}H_k^{\text{uf}}(G;\mathbb{R}/\mathbb{Z}) &\cong H^{n-k}(G; \ell^\infty(G, \mathbb{R}/\mathbb{Z})) \\ &\cong H^{n-k}(G; \operatorname{Hom}(\mathbb{Z}G, \mathbb{R}/\mathbb{Z})) \\ &= 0.\end{aligned}$$

The last equality follow because $\operatorname{Hom}(\mathbb{Z}G; \mathbb{R}/\mathbb{Z})$ is an injective G-module [15, Proposition 6.1]. □

6.6 Uniformly Finite Homology and Amenable Groups

In this section we present the main result of joint work with Francesca Diana about uniformly finite homology of amenable groups [9]. We will only give a summary here, for a more detailed treatment confer the article. If G is an amenable group, write $LM(G) \subset \ell^\infty(G)'$ for the space of left invariant means and $M(G)$ for the space of bi-invariant means on G.

We will show that uniformly finite homology groups of amenable groups are often infinite dimensional. Our principal observation is the following: Let G be an infinite amenable group. On the one hand, by Proposition 6.2.1 every mean m on G induces a left inverse

$$\begin{aligned} m_*\colon H_*(G;\ell^\infty(G)) &\longrightarrow H_*(G;\mathbb{R}) \\ [c\otimes\varphi] &\longmapsto m(\varphi)\cdot[c] \end{aligned}$$

to the comparison map. On the other hand, there are many candidates for these splitting maps, since by the following classical result of Chou [28, Theorem 1] there are many different invariant means:

Theorem 6.6.1. *If G is an infinite amenable group, then G has exactly $2^{2^{|G|}}$ left invariant means, where $|G|$ denotes the cardinality of G. Thus $LM(G)$ is infinite dimensional.*

The main idea is to use the maps induced by different means to differentiate between different classes. In general, however, there might not be enough *cycles* to be separated by means, e.g. $H_n^{\mathrm{uf}}(\mathbb{Z}^n;\mathbb{R})\cong\mathbb{R}$ and all the functions $m_n\colon H_n^{\mathrm{uf}}(\mathbb{Z}^n;\mathbb{R})\longrightarrow H_n(\mathbb{Z}^n;\mathbb{R})$ induced by means coincide. The problem is that for a cycle $c\in C_n(G;\mathbb{R})$ and a bounded function $\varphi\in\ell^\infty(G;\mathbb{R})$, the chain $c\otimes\varphi\in C_n(G;\ell^\infty(G;\mathbb{R}))$ is not necessarily a cycle since the G-action on $\ell^\infty(G;\mathbb{R})$ is not trivial.

We avoid this issue by considering cycles $c\in C_n(G;\mathbb{R})$ that are supported in a subgroup $H\leq G$ and functions $\varphi\in\ell^\infty(G;\mathbb{R})$ that are invariant with respect to the action of H. Then $c\otimes\varphi\in C_n(G;\ell^\infty(G;\mathbb{R}))$ will be a cycle as well.

The next step is to show that, if $H\leq G$ has *infinite index*, then we can find an infinite family of different G-invariant means that can be separated by *H-invariant functions*, a result that might be of independent interest:

Theorem 6.6.2 ([9, Theorem 3.11])**.** *Let G be a finitely generated amenable group and $H\leq G$ a subgroup such that $[G:H]=\infty$. Then there exists an infinite family $(m_j)_{j\in J}$ of left G-invariant means and an infinite family $(f_j)_{j\in J}$ of (left) H-invariant functions in $\ell^\infty(G)$, such that $m_k(f_j)=\delta_{k,j}$ for any $k,j\in J$.*

Sketch of proof. We can inductively construct a family $(A_k^l)_{k,l\in\mathbb{N}}$ (Figure 6.3) of subsets in G such that:

- For each $(k,l)\in\mathbb{N}^2$, the set A_k^l is a (right-translated) l-Ball, i.e., there exists a $g\in G$ such that
$$A_k^l = B_l(e)\cdot g.$$
- Their orbits under left-translation by elements in H are disjoint, i.e., for each pair of distinct indices $(k,l)\neq(k',l')\in\mathbb{N}^2$
$$\bigl(H\cdot A_k^l\bigr)\cap\bigl(H\cdot A_{k'}^{l'}\bigr)=\emptyset.$$

For each $k\in\mathbb{N}$ set
$$T^k := \bigcup_{l\in\mathbb{N}} H\cdot A_k^l.$$
The family $(T^k)_{k\in\mathbb{N}}$ is pairwise disjoint and the corresponding characteristic functions χ_{T^k} are H-invariant for each $k\in\mathbb{N}$. Using the correspondence between Følner-sequences and means, we can construct a sequence $(m_j)_{j\in\mathbb{N}}$ of G-invariant means such that m_j is supported in T^j. $\square$

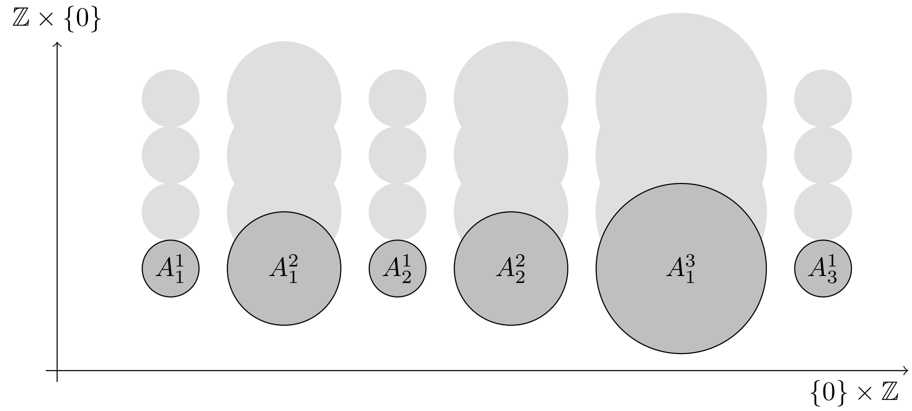

Figure 6.3: Constructing a family of disjoint $(\mathbb{Z} \times \{0\})$-invariant subsets supporting invariant means in $\mathbb{Z}^2$

Hence, if there is a cycle $c \in C_n(H;\mathbb{R})$ such that $[i_n(c)] \in H_n(G;\mathbb{R})$ is not trivial, then $([i(c) \otimes f_j])_{j \in J}$ is a family of linear independent classes in $H_n^{\mathrm{uf}}(G;\mathbb{R})$ (with f_j as in Theorem 6.6.2 and $i\colon H \hookrightarrow G$ the canonical inclusion). Therefore, we have proven:

Theorem 6.6.3 ([9, Theorem 3.8])**.** *Let G be a finitely generated amenable group. Let $H \leq G$ be an infinite index subgroup. For each $n \in \mathbb{N}$ such that the map*

$$H_n(i;\mathbb{R})\colon H_n(H;\mathbb{R}) \longrightarrow H_n(G;\mathbb{R})$$

induced by the inclusion $i\colon H \hookrightarrow G$ is non-trivial, $\dim_{\mathbb{R}} H_n^{\mathrm{uf}}(G;\mathbb{R}) = \infty$ *holds.*

There are trivial reasons for uniformly finite homology to be finite dimensional that we will also see in some of our examples:

Lemma 6.6.4. Let G be virtually an n-dimensional $\mathbb{R}$-Poincaré duality group. Then $H_k^{\mathrm{uf}}(G;\mathbb{R}) = 0$ for all $k \in \mathbb{N}_{>n}$ and $H_n^{\mathrm{uf}}(G;\mathbb{R}) \cong \mathbb{R}$.

Proof. Since uniformly finite homology is a quasi-isometry invariant, after passing to a finite index subgroup, we can assume that G is a Poincaré duality group. The vanishing of homology in higher degrees is part of the definition of a Poincaré duality group. In the top dimensional case we have

$$H_n^{\mathrm{uf}}(G;\mathbb{R}) \cong H_n(G;\ell^\infty(G)) \cong H^0(G;\ell^\infty(G)) \cong \ell^\infty(G)^G \cong \mathbb{R}. \qquad \square$$

We proceed to discuss several applications of Theorem 6.6.3. We start with the situation in degree 1:

Corollary 6.6.5. Let G be a finitely generated amenable group, such that the first homology group $H_1(G;\mathbb{R})$ is non-trivial, i.e., that the abelianization of G is not a torsion group. Then

$$\dim_{\mathbb{R}} H_1^{\mathrm{uf}}(G;\mathbb{R}) = \begin{cases} 1 & \text{if } G \text{ is virtually } \mathbb{Z} \\ \infty & \text{otherwise.} \end{cases}$$

Sketch of Proof. Pick any $g \in G$ such that the image of g in G_{ab} has infinite order. Either $H := \langle g \rangle \leq G$ has finite index and G is virtually $\mathbb{Z}$ or H satisfies the conditions of Theorem 6.6.3 in degree 1 by the naturality of the isomorphism $H_1(G;\mathbb{R}) \cong G_{\mathrm{ab}} \otimes \mathbb{R}$. $\square$

Corollary 6.6.6 ([9, Example 4.4])**.** Let G be a finitely generated virtually nilpotent group of Hirsch rank $h \in \mathbb{N}$. Then

$$\dim_{\mathbb{R}} H_k^{\mathrm{uf}}(G;\mathbb{R}) = \begin{cases} 1 & \text{if } k = h \\ \infty & \text{if } k \in \{0,\dots,h-1\} \\ 0 & \text{else.} \end{cases}$$

Sketch of Proof. Using the calculation of the homology groups of finitely generated nilpotent groups by Baumslag, Miller and Short [4], we can show that G has an infinite index subgroup $H \leq G$ satisfying the conditions in Theorem 6.6.3 for $k \in \{0,\dots,h-1\}$. The other cases follow because G is virtually a Poincaré duality group. $\square$

Example 6.6.7 ([9, Example 4.2])**.** Let G, H be finitely generated infinite amenable groups, such that that $H_k(H;\mathbb{R}) \neq 0$ for a $k \in \mathbb{N}$. Then

$$\dim_{\mathbb{R}} H_k^{\mathrm{uf}}(G \rtimes H;\mathbb{R}) = \infty.$$

In particular, for all $l \in \mathbb{N}$

$$\dim_{\mathbb{R}} H_k^{\mathrm{uf}}(\mathbb{Z}^l;\mathbb{R}) = \begin{cases} 1 & \text{if } k = l \\ \infty & \text{if } k \in \{0,\dots,l-1\} \\ 0 & \text{else.} \end{cases}$$

Example 6.6.8 ([9, Example 4.5])**.** Consider $A \in \mathrm{SL}(2,\mathbb{Z})$. Then for the semidirect product $\mathbb{Z}^2 \rtimes_A \mathbb{Z}$ given by the action of $\mathbb{Z}$ on $\mathbb{Z}^2$ induced by A, we have

$$\dim_{\mathbb{R}} H_k^{\mathrm{uf}}(\mathbb{Z}^2 \rtimes_A \mathbb{Z};\mathbb{R}) = \begin{cases} 1 & \text{if } k = 3 \\ \infty & \text{if } k \in \{0,1,2\} \\ 0 & \text{else.} \end{cases}$$

In particular, this example includes cocompact lattices in Sol [70] and the integral three-dimensional Heisenberg group.

Using the work of Fujiwara and Oshika on the bounded cohomology of 3-manifold groups and our calculation in the amenable case, we get:

Theorem 6.6.9. *Let M be a closed irreducible 3-manifold with fundamental group $G := \pi_1(M)$. Then either G is finite or*

$$\dim_{\mathbb{R}} H_2^{\mathrm{uf}}(G;\mathbb{R}) = \infty.$$

Proof. By the calculation of the second bounded cohomology of 3-manifolds of Fujiwara and Oshika [41], either $H_b^2(G;\mathbb{R})$ is infinite dimensional or G is virtually solvable. In the first case, the space of quasi-morphism $Q(G)$ is infinite dimensional and so is $H^1_{\mathrm{uf}}(G;\mathbb{R})$ by Proposition 6.4.3. By assumption and the sphere theorem [67, 77], after possibly passing to an orientable cover, we can assume that M is aspherical, thus G is virtually a Poincaré duality group. Hence, also $H_2^{\mathrm{uf}}(G;\mathbb{R})$ is infinite dimensional. Assume now that G is virtually solvable. By the classification of solvable 3-manifold groups by Evans and Moser [38], G is finite or virtually the fundamental group of a torus bundle over S^1. Thus if G is infinite, $H_2^{\mathrm{uf}}(G;\mathbb{R})$ is infinite dimensional by Example 6.6.8. $\square$

6.7 Degree Zero

In this section we will study 0-classes in uniformly finite homology of amenable groups. We will distinguish between classes that can be detected by means and classes that are "*mean-invisible*" and will show that both classes correspond to infinite dimensional spaces if the group is infinite. In particular, $H_0^{\mathrm{uf}}(G;\mathbb{R})$ is infinite dimensional if G is an infinite amenable group. This section is based on joint work with Francesca Diana [9].

Definition 6.7.1. Let G be a finitely generated group. We call the subspace

$$\widehat{H}_0(G;\ell^\infty(G)) := \bigl\{c \in H_0(G;\ell^\infty(G)) \bigm| \forall_{m\in M(G)} \quad m_0(c) = 0\bigr\}$$

the *mean-invisible part of* $H_0(G;\ell^\infty(G))$. Recall that for $m \in M(G)$ we denote by m_0 the function on $H_0(G;\ell^\infty(G)) \cong \ell^\infty(G)_G$ induced by m.

6.7.1 Classes Visible to Means

Since for an infinite amenable group G there are infinitely many different means, there are infinitely many classes in $H_0(G;\ell^\infty(G))$ that *can* be detected by invariant means:

Proposition 6.7.2 ([9, Theorem 3.7])**.** *Let G be a finitely generated amenable group. Then*

$$\dim_{\mathbb{R}} H_0(G;\ell^\infty(G))/\widehat{H}_0(G;\ell^\infty(G)) = \infty.$$

Proof. By the definition of $\widehat{H}_0(G;\ell^\infty(G))$, there is a linear injection of the space of invariant means into the following dual space:

$$\begin{aligned} LM(G) &\longhookrightarrow \bigl(H_0(G;\ell^\infty(G))/\widehat{H}_0(G;\ell^\infty(G))\bigr)' \\ m &\longmapsto \bigl([\varphi\otimes g] \longmapsto m(\varphi)\bigr). \end{aligned}$$

Therefore, by Theorem 6.6.1, the dual space $(H_0(G;\ell^\infty(G))/\widehat{H}_0(G;\ell^\infty(G)))'$ is infinite dimensional, and hence also $H_0(G;\ell^\infty(G))/\widehat{H}_0(G;\ell^\infty(G))$. $\square$

Next, we will show that the space we have considered in Proposition 6.7.2 can be identified with reduced uniformly finite homology in degree 0, which we will define below. To demonstrate this, we will prove a result about semi-norms on $H_0(G;\ell^\infty(G))$ that might be of independent interest:

Definition 6.7.3. Let G be an amenable group. The ℓ^∞-norm on $\ell^\infty(G)$ and the semi-norm

$$\begin{aligned} \|\cdot\|_\mu \colon \ell^\infty(G) &\longrightarrow \mathbb{R} \\ x &\longmapsto \sup_{m\in M(G)} |m(x)|, \end{aligned}$$

induce two semi-norms on $H_0(G;\ell^\infty(G))$ that we will also denote by $\|\cdot\|_\infty$ and $\|\cdot\|_\mu$. Since every mean vanishes on $\operatorname{im}\partial_1^{\mathrm{uf}}$, we have $\|\varphi\|_\mu = \|[\varphi]\|_\mu$ for all $\varphi \in C_0^{\mathrm{uf}}(G;\mathbb{R})$.

Proposition 6.7.4. *Let G be an amenable group. Then the semi-norms $\|\cdot\|_\infty$ and $\|\cdot\|_\mu$ on $H_0(G;\ell^\infty(G))$ are equivalent.*

Proof. Clearly, $(\ell^\infty(G)_G)' \cong B_G(\ell^\infty(G),\mathbb{R})$ and we will identify these spaces here. Every functional in $B_G(\ell^\infty(G),\mathbb{R})$ can be written as the difference of two positive functionals in $B_G(\ell^\infty(G),\mathbb{R})$ [69, Proposition 2.2]. The claim about the norms is then a classical result, we recall a proof from [50, Theorem 23.2]:

Clearly, $\|\cdot\|_\mu \leq \|\cdot\|_\infty$ on $\ell^\infty(G)$ and hence also on $H_0(G;\ell^\infty(G))$. We write $P_1 := \{\lambda\cdot m \mid m \in M(G),\ \lambda\in[0,1]\}$ for the set of positive functionals of norm smaller or equal 1 in $B_G(\ell^\infty(G),\mathbb{R})$. As we have noted above, every function $f \in (\ell^\infty(G)_G)' \cong B_G(\ell^\infty(G),\mathbb{R})$ can be written as

$$f = r\cdot(\varphi_1-\varphi_2)$$

with $r\in\mathbb{R}_{>0}$ and $\varphi_1,\varphi_2\in P_1$. Since $\{\varphi_1-\varphi_2 \mid \varphi_1,\varphi_2\in P_1\}$ is weak-$*$-closed and convex, by the weak-$*$-compactness of the unit ball, there exists a constant $R\in\mathbb{R}_{>0}$ such that every $f\in(\ell^\infty(G)_G)'$ with $\|f\|_\infty\leq 1$ can be written as

$$f = R\cdot(\varphi_1-\varphi_2).$$

By the Hahn-Banach-Theorem, we have for all $x\in\ell^\infty(G)_G$

$$\|x\|_\infty = \sup\bigl\{|f(x)| \bigm| f\in(\ell^\infty(G)_G)',\ \|f\|_\infty\leq 1\bigr\},$$

hence

$$\begin{aligned}\|x\|_\infty &\leq \sup\{|R\cdot(\varphi_1(x)-\varphi_2(x))| \mid \varphi_1,\varphi_2\in P_1\}\\ &\leq 2\cdot R\cdot\sup\{|\varphi(x)| \mid \varphi\in P_1\}\\ &= 2\cdot R\cdot\sup\{|m(x)| \mid m\in M(G)\}\\ &= 2\cdot R\cdot\|x\|_\mu.\end{aligned}$$

□

Proposition 6.7.4 implies directly the following correspondence between classes that can be detected by means and reduced uniformly finite homology, that was told to us by Michał Marcinkowski:

Corollary 6.7.5 ([69, Proposition 2.1])**.** Let G be an amenable group. Then

$$\overline{\{v-g\cdot v \mid v\in\ell^\infty(G),\ g\in G\}}^{\|\cdot\|_{\ell^\infty}} = \{c\in\ell^\infty(G) \mid \forall_{m\in M(G)}\quad m(c)=0\}.$$

In other words, if

$$\overline{H}_0^{\mathrm{uf}}(G;\mathbb{R}) := C_0^{\mathrm{uf}}(G;\mathbb{R})/\overline{\operatorname{im}\partial_1^{\mathrm{uf}}}^{\|\ \|_{\ell^\infty}}$$

is the *reduced 0th-uniformly finite homology group*, we have

$$\overline{H}_0^{\mathrm{uf}}(G;\mathbb{R}) \cong H_0(G;\ell^\infty(G))/\widehat{H}_0(G;\ell^\infty(G)).$$

In particular, $\overline{H}_0^{\mathrm{uf}}(G;\mathbb{R})$ is infinite dimensional if G is an infinite amenable group.

Proof. An element $\varphi \in \ell^\infty(G)$ is in $\{c \in \ell^\infty(G) \mid \forall_{m \in M(G)} \quad m(c) = 0\}$ if and only if $\|[\varphi]\|_\mu = \|\varphi\|_\mu = 0$, hence if and only if $\|[\varphi]\|_\infty = 0$. Therefore

$$\overline{\{v - g \cdot v \mid v \in \ell^\infty(G),\ g \in G\}}^{\|\ \|_{\ell^\infty}} = \{c \in \ell^\infty(G) \mid \forall_{m \in M(G)} \quad m(c) = 0\}.$$

That $\overline{H}_0^{\mathrm{uf}}(G;\mathbb{R})$ is infinite dimensional follows now from Proposition 6.7.2. □

6.7.2 Distinguishing Classes by Asymptotic Behaviour

We will reformulate White's criterion for the vanishing of 0-degree uniformly finite homology classes, Corollary 6.5.5, in terms of the asymptotic behaviour of chains in degree 0 with respect to the behaviour of a Følner-sequence. We will first need a notion to compare the asymptotic behaviour of functions:

Definition 6.7.6. Let $\alpha, \beta \colon \mathbb{N} \longrightarrow \mathbb{R}_{\geq 0}$ be two functions.

(i) We say β *dominates* α and write $\alpha \prec \beta$ if $\beta(n)$ is zero for at most finitely many $n \in \mathbb{N}$ and
$$\lim_{n \to \infty} \frac{\alpha(n)}{\beta(n)} = 0.$$

(ii) We write $\alpha \sim \beta$ if $\beta(n)$ is zero for at most finitely many $n \in \mathbb{N}$ and
$$\lim_{n \to \infty} \frac{\alpha(n)}{\beta(n)} \in \mathbb{R}_{>0},$$
or if both $\alpha(n)$ and $\beta(n)$ are zero for almost all $n \in \mathbb{N}$.

(iii) Finally, we also write $\alpha \preceq \beta$ if $\alpha \prec \beta$ or $\alpha \sim \beta$.

Definition 6.7.7. Let G be a finitely generated group with a fixed word metric. A *Følner sequence in* G is a sequence $(S_j)_{j \in \mathbb{N}}$ of non-empty finite subsets of G such that for each $r \in \mathbb{N}_{>0}$
$$\lim_{j \to \infty} \frac{|\partial_r(S_j)|}{|S_j|} = 0.$$

Definition 6.7.8. Let $S := (S_j)_{j \in \mathbb{N}}$ be a Følner-sequence in G. Consider the sequence of (not necessarily invariant) means $(m_j)_{j \in \mathbb{N}}$ given by
$$\begin{aligned} m_j \colon \ell^\infty(G) &\longrightarrow \mathbb{R} \\ \varphi &\longmapsto \frac{1}{|F_j|} \sum_{g \in F_j} \varphi(g). \end{aligned}$$

We call S *simple* if $(m_j)_{j \in \mathbb{N}}$ weak-$*$-converges in $\ell^\infty(G)'$. In this case, write m_S for the weak-$*$-limit of $(m_j)_{j \in \mathbb{N}}$. The limit m_S is always an invariant mean [26, Theorem 4.9.2].

Since the space of means is weak-$*$-compact, every Følner sequence has a simple subsequence.

Definition 6.7.9. Let G be a finitely generated group and $S := (S_j)_{j\in\mathbb{N}}$ a Følner sequence for G. For all $c \in \ell^\infty(G)$ we define a function

$$\begin{aligned} \beta_c^S \colon \mathbb{N} &\longrightarrow \mathbb{R} \\ n &\longmapsto \frac{|\sum_{s\in S_n} c(s)|}{|S_n|}. \end{aligned}$$

Finally, we also consider the asymptotic behaviour of the boundaries given by the function

$$\begin{aligned} \sigma_S \colon \mathbb{N} &\longrightarrow \mathbb{R} \\ n &\longmapsto \frac{|\partial_1 S_n|}{|S_n|}. \end{aligned}$$

Lemma 6.7.10 ([9, Lemma 5.6])**.** Let G be a finitely generated amenable group and S a Følner sequence in G. Consider $c \in \ell^\infty(G)$.

(i) We have $0 \preceq \beta_c^S \preceq 1$.

(ii) If S is simple, then $\beta_c^S \sim 1$ if and only if $m_S(c) \neq 0$. In particular, $\beta_c^S \sim 1$ implies that there is a simple subsequence S' of S such that $m_{S'}(c) \neq 0$.

(iii) If $\beta_c^S \succ \sigma_S$, then $[c] \neq 0 \in H_0^{\mathrm{uf}}(G;\mathbb{R})$.

(iv) Conversely, if $[c] \neq 0 \in H_0^{\mathrm{uf}}(G;\mathbb{R})$ then there exists a Følner sequence S' such that $\sigma_{S'} \prec \beta_c^{S'}$.

(v) More generally: If $(c_n)_{n\in\mathbb{N}}$ is a sequence in $\ell^\infty(G)$ such that $\sigma_S \prec \beta_{c_0}^S$ and

$$\forall_{n\in\mathbb{N}} \quad \beta_{c_n}^S \prec \beta_{c_{n+1}}^S,$$

then the family $([c_n])_{n\in\mathbb{N}}$ of classes in $H_0^{\mathrm{uf}}(G;\mathbb{R})$ is linearly independent.

Proof.

(i) This is clear by the definition of β_c^S.

(ii) If S is simple, the sequence $(|\sum_{s\in S_n} c(s)|/|S_n|)_{n\in\mathbb{N}}$, that we have to consider to decide whether $\beta_c^S \sim 1$, converges to $|m_S(c)|$.

(iii) Assume $[c] = 0$. By Whyte's criterion, Corollary 6.5.5, there exist $C, r \in \mathbb{N}_{>0}$ such that for all $n \in \mathbb{N}$

$$\frac{|\partial_1 S_n|}{\left|\sum_{s\in S} c(s)\right|} \geq \frac{1}{|B_r(e)|} \cdot \frac{|\partial_r S_n|}{\left|\sum_{s\in S} c(s)\right|} \geq \frac{1}{C \cdot |B_r(e)|} > 0$$

Hence $\beta_c^S \not\succ \sigma_S$.

(iv) By Whyte's criterion, if $[c] \neq 0$, for each $n \in \mathbb{N}$ there exists a finite subset $S_n' \subseteq G$ such that

$$n \cdot |\partial_1 S_n'| < \Big|\sum_{s\in S_n'} c(s)\Big|;$$

hence

$$\frac{1}{\|c\|_\infty} \cdot \frac{|\partial_1 S'_n|}{|S'_n|} \leq \frac{|\partial_1 S'_n|}{\left|\sum_{s \in S'_n} c(s)\right|} < \frac{1}{n}.$$

In particular $S' := (S'_n)_{n\in\mathbb{N}}$ is a Følner sequence and $\sigma_{S'} \prec \beta_c^{S'}$.

(v) For each $c \in \ell^\infty(G)$, such that $\sum_{s\in S_n} c(s)$ is zero for at most finitely many $n \in \mathbb{N}$, we define a subspace

$$C_0^{\mathrm{uf}}(G;\mathbb{R})^{\preceq c} := \left\{ c' \in C_0^{\mathrm{uf}}(G;\mathbb{R}) \;\middle|\; \lim_{n\to\infty} \frac{\sum_{s\in S_n} c'(s)}{\sum_{s\in S_n} c(s)} \text{ exists} \right\}.$$

Choose a splitting of $\mathbb{R}$-vector spaces $C_0^{\mathrm{uf}}(G;\mathbb{R}) = C_0^{\mathrm{uf}}(G;\mathbb{R})^{\preceq c} \oplus V$ and define a linear function

$$\begin{aligned} \gamma_c^S \colon C_0^{\mathrm{uf}}(G;\mathbb{R}) = C_0^{\mathrm{uf}}(G;\mathbb{R})^{\preceq c} \oplus V &\longrightarrow \mathbb{R} \\ (c', c'') &\longmapsto \lim_{n\to\infty} \frac{\sum_{s\in S_n} c'(s)}{\sum_{s\in S_n} c(s)}. \end{aligned}$$

Let $a \in C_0^{\mathrm{uf}}(G;\mathbb{R})$ be a boundary. Then the function β_a^S/σ_S is bounded by Whyte's criterion, so $\beta_a^S \prec \sigma_S$. Thus if $\sigma_S \prec \beta_c^S$ then $\beta_a^S \prec \beta_c^S$ and hence $\gamma_c^S(a) = 0$. Therefore, γ_c^S induces a linear map

$$\overline{\gamma_c^S} \colon H_0^{\mathrm{uf}}(G;\mathbb{R}) \longrightarrow \mathbb{R}.$$

In our situation we have $\gamma_{c_i}^S(c_j) = \delta_{ij}$ for all $j \leq i$ in $\mathbb{N}$. Hence, $([c_n])_{n\in\mathbb{N}}$ is linearly independent. □

Notation 6.7.11. *For each $k \in \mathbb{N}$ set $[n^k] := \{m^k \mid m \in \mathbb{N}\}$.*

Example 6.7.12. Consider the Følner sequence B in $\mathbb{Z}$ given by the sequence of balls $(\{-n, -n+1, \dots, n\})_{n\in\mathbb{N}}$. Then for each $k \in \mathbb{N}_{>0}$

$$\beta_{[n^k]}^B \sim \bigl(m \longmapsto m^{1-k}\bigr).$$

In particular, the sequence $\bigl([n^k]\bigr)_{k\in\mathbb{N}_{>0}}$ of subsets in $\mathbb{Z}$ satisfies the conditions in Lemma 6.7.10 (v) and hence induces a sequence of linearly independent classes in $H_0^{\mathrm{uf}}(\mathbb{Z};\mathbb{R})$.

Different Følner sequences can of course detect different classes:

Example 6.7.13. Consider the Følner sequence T in $\mathbb{Z}$ given by the sequence of balls $(\{-n-1, -n, \dots, -1\})_{n\in\mathbb{N}}$. Then $\beta_{[n^k]}^T \sim 0$ for each $k \in \mathbb{N}_{>0}$.

6.7.3 Sparse Classes

We now introduce a geometric condition for a subset to be mean-invisible:

Definition 6.7.14. Let G be a finitely generated group. We call a subset $\Gamma \subseteq G$ *(asymptotically) sparse* (Figure 6.4) if

$$\exists_{C\in\mathbb{N}} \quad \forall_{r\in\mathbb{N}_{>0}} \quad \exists_{R\in\mathbb{N}_{>0}} \quad \forall_{g\in G\setminus B_R(e)} \quad |\Gamma \cap B_r(g)| \leq C.$$

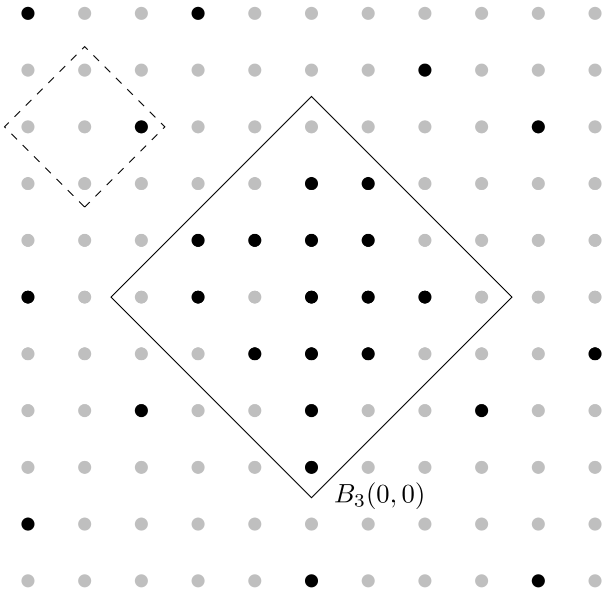

Figure 6.4: The black dots indicate a sparse subset in $\mathbb{Z}^2$ with the standard word metric.

Example 6.7.15. For each $k \in \mathbb{N}_{>1}$ the subsets $\{n^k \mid n \in \mathbb{N}\} \subseteq \mathbb{Z}$ are sparse.

Lemma 6.7.16 ([9, Lemma 5.10])**.** Let G be an infinite, finitely generated amenable group and $\Gamma \subseteq G$ a sparse subset. Then we have

$$[\chi_\Gamma] \in \widehat{H}_0^{\mathrm{uf}}(G;\mathbb{R}).$$

Proof. Let $C \in \mathbb{N}$ be a constant for Γ as in the definition of sparse. Let m be a left-invariant mean on G. Consider $r \in \mathbb{N}_{>0}$. Since G is infinite we have for all $R \in \mathbb{N}_{>0}$ that $m(\chi_\Gamma) = m(\chi_{\Gamma \setminus B_R(e)})$. Hence, taking $R \in \mathbb{R}_{>0}$ as in the definition of sparse and possibly replacing Γ with $\Gamma \setminus B_R(e)$, we can assume that for all $g \in G$

$$|\{\gamma \in \Gamma \mid \exists_{h \in B_r(e)} \quad \gamma \cdot h = g\}| = |\Gamma \cap B_r(g)| \leq C.$$

Therefore the coefficients of $\sum_{h \in B_r(e)} \chi_{\Gamma \cdot h}$ are bounded by the constant C. So we see that

$$|B_r(e)| \cdot m(\chi_\Gamma) = \sum_{h \in B_r(e)} m(\chi_{\Gamma \cdot h}) = m\left(\sum_{h \in B_r(e)} \chi_{\Gamma \cdot h}\right) \leq C.$$

Hence $m(\chi_\Gamma) = 0$. □

6.7.4 Constructing Sparse Classes

We recall the notion of tilings, which will be the building blocks for our sparse classes:

Definition 6.7.17. Let G be a finitely generated group with a word metric. For $r \in \mathbb{N}_{>0}$ we call a subset $T \subseteq G$ an *r-tiling for G* if

(i) $\forall_{g_1,g_2\in T} \quad B_r(g_1) \cap B_r(g_2) \neq \emptyset \Longrightarrow g_1 = g_2$

(ii) $G = \bigcup_{g\in T} B_{2\cdot r}(g)$.

By Zorn's Lemma, for all $r \in \mathbb{N}_{>0}$ there exists an r-tiling [26, Proposition 5.6.3].

Lemma 6.7.18 ([26, Proposition 5.6.4])**.** Let G be a finitely generated amenable group and $(S_j)_{j\in\mathbb{N}}$ a Følner sequence. Let $r \in \mathbb{N}_{>0}$ and let T be an r-tiling for G. Set

$$T_j := \{g \in T \mid B_r(g) \subseteq S_j\} \subseteq S_j.$$

Then there exists an $l(T) \in \mathbb{N}$ such that for all $j \geq l(T)$

$$\frac{1}{2\cdot|B_{2\cdot r}(e)|} \leq \frac{|T_j|}{|S_j|}.$$

Sketch of proof. One obviously gets the corresponding estimate for the points in the $(2\cdot r)$-interior of S_j. But the number of points in the r-boundary of S_j relative to $|S_j|$ goes to zero for $j \to \infty$. □

The next theorem tells us that we can realise any non-trivial asymptotic behaviour by a sparse set. This result is a stronger version of [9, Theorem 5.1] and the proof is simpler.

Theorem 6.7.19. *Let G be a finitely generated infinite amenable group with a word metric. Then there is a Følner sequence S in G such that for each growth function $c\colon \mathbb{N} \longrightarrow \mathbb{R}_{>0}$ such that $c \prec 1$, there is a sparse subset $\Gamma \subset G$ such that $\beta_\Gamma^S \sim c$. In particular, there is an uncountable family of linear independent sparse classes in $H_0^{\mathrm{uf}}(G;\mathbb{R})$.*

Proof. Choose an r-tiling T^r for all $r \in \mathbb{N}_{>0}$. Let S be a Følner sequence such that for all $j \in \mathbb{N}$

$$d\Big(\bigcup_{l=0}^{j} S_l, S_{j+1}\Big) > j. \tag{$*$}$$

For instance, start with an arbitrary Følner sequence and inductively translate the j-th set away from the first $j-1$ sets. Let $c\colon \mathbb{N} \longrightarrow \mathbb{R}_{>0}$ be a growth function with $c \prec 1$. Without changing the asymptotic behaviour of c, we can assume that $c(r) \leq |S_r|$ for all $r \in \mathbb{N}$. For each $r \in \mathbb{N}_{>0}$ define a number

$$\rho(r) := \max\Big\{r' \in \mathbb{N} \;\Big|\; c(r) \leq \frac{|T_r^{r'}|}{|S_r|}\Big\} \in \mathbb{N}_{>0}.$$

Since $c \prec 1$, i.e., $\lim_{r\to\infty} c(r) = 0$, for all $r' \in \mathbb{N}$ we can find an $R \in \mathbb{N}_{>0}$ such that for all $r \in \mathbb{N}_{\geq R}$ we have

$$c(r) < \frac{1}{2\cdot|B_{2\cdot r'}(e)|}$$

and

$$r \geq \max\{l(T^0), \ldots, l(T^{r'})\}.$$

Therefore, by Lemma 6.7.18 we see that $\lim_{r\to\infty} \rho(r) = \infty$. We have

$$c(r) \leq \frac{|T_r^{\rho(r)}|}{|S_r|},$$

hence we can find a subset $L_r \subset T_r^{\rho(r)}$, such that

$$c(r) \leq \frac{|L_r|}{|S_r|} \leq c(r) + \frac{1}{|S_r|}. \qquad (**)$$

Set $\Gamma := \bigcup_{r\in\mathbb{N}_{>0}} L_r$. By $(**)$ we have $c \sim \beta_\Gamma^S$. By $(*)$ and the tiling condition, we see that Γ is sparse.

For the last claim, set for instance for each $\alpha \in (0,1]$

$$\begin{aligned} \sigma_S^\alpha \colon \mathbb{N} &\longrightarrow \mathbb{R}_{>0} \\ n &\longmapsto \sigma_S(n)^\alpha. \end{aligned}$$

then for each pair $\alpha < \beta \in (0,1]$ we have $\sigma_S^\alpha \prec \sigma_S^\beta$ and by Lemma 6.7.10, the family of sparse classes in $H_0^{\mathrm{uf}}(G;\mathbb{R})$ corresponding to $(\sigma_S^\alpha)_{\alpha\in(0,1)}$ by the above construction is linearly independent. $\square$

Corollary 6.7.20. The space $\widehat{H}_0^{\mathrm{uf}}(G;\mathbb{R})$ is infinite dimensional if G is a finitely generated infinite amenable group.

6.7.5 Sparse Classes and the Cross Product

There is no abundance of tools to calculate uniformly finite homology (or other "exotic" homology theories). Künneth-type formulas for uniformly finite homology, as studied by Francesca Diana in her thesis, are therefore very interesting. In particular, Diana has shown [30, Theorem 3.1.3], that the cross product map is injective for reduced uniformly finite homology (i.e., without the mean-invisible part), if at least one of the groups is amenable. Our next proposition together with Example 6.7.12 shows that this cannot be generalised to the unreduced case. Let us first recall the definition of the cross product in uniformly finite homology [30, Definition 3.1.1]:

Definition 6.7.21 (Cross Product in Degree 0)**.** Let X and Y be metric spaces. Let R be a normed ring, i.e., a normed Abelian group with a compatible (unital) ring structure, such that the norm is multiplicative. For each $n \in \mathbb{N}$, there is an R-morphism

$$\begin{aligned} \times\colon C_0^{\mathrm{uf}}(X;R) \otimes C_n^{\mathrm{uf}}(Y;R) &\longrightarrow C_n(X\times Y;R) \\ \sum_{x\in X} c(x)\cdot x \otimes \sum_{y\in Y^{n+1}} a(y)\cdot y &\longmapsto \sum_{x\in X,\ y\in Y^{n+1}} a(y)\cdot c(x)\cdot \big((x,y_0),\dots,(x,y_0)\big). \end{aligned}$$

This induces a well-defined R-map in uniformly finite homology

$$\times\colon H_0^{\mathrm{uf}}(X;R) \otimes H_n^{\mathrm{uf}}(Y;R) \longrightarrow H_n^{\mathrm{uf}}(X\times Y;\mathbb{R}),$$

called the *cross product in uniformly finite homology.*

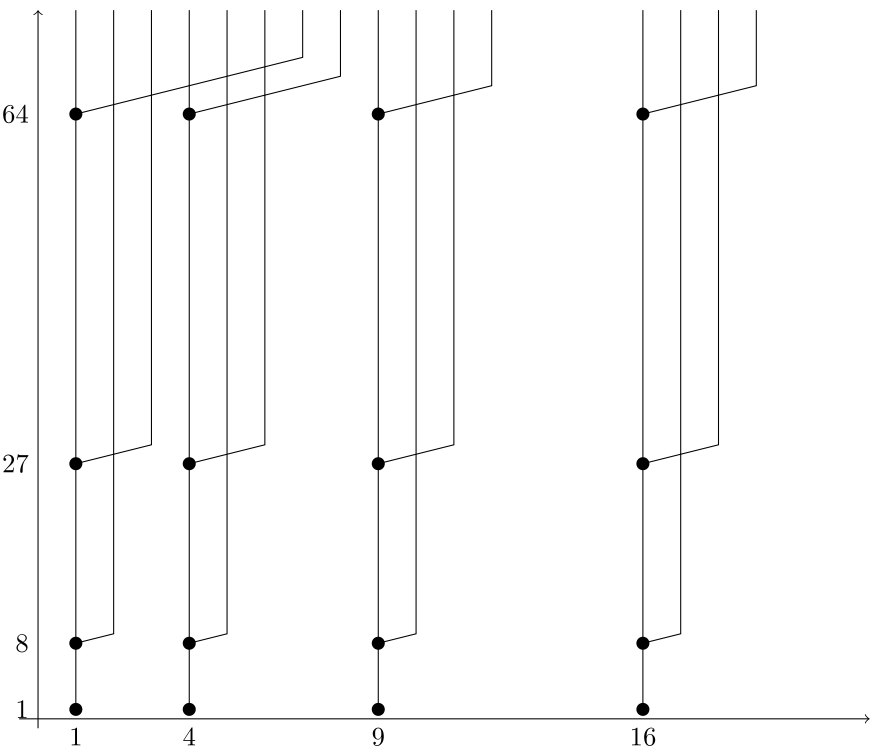

Figure 6.5: The class of $[n^2] \times [n^3]$ is trivial in $H_0^{\mathrm{uf}}(\mathbb{Z}^2; \mathbb{R})$

Proposition 6.7.22. *For each $(k, l) \in \mathbb{N} \times \mathbb{N}$ such that $k \cdot l > 4$, the class in $H_0^{\mathrm{uf}}(\mathbb{Z}^2; \mathbb{Z})$ corresponding to $[n^k] \times [n^l]$ is trivial. In particular, the cross product map*

$$\times \colon H_0^{\mathrm{uf}}(\mathbb{Z}; \mathbb{Z}) \otimes H_0^{\mathrm{uf}}(\mathbb{Z}; \mathbb{Z}) \longrightarrow H_0^{\mathrm{uf}}(\mathbb{Z}^2; \mathbb{Z})$$

is not injective.

Proof. To simplify notation, we consider only $k = 3 = l$, the proof is the same in the other cases. Consider the word metric d on $\mathbb{Z}^2$ corresponding to the generating set $\{(0,1), (1,0), (1,1)\}$. Intuitively, for each point in $[n^3] \times [n^3]$ we will construct a ray in the corresponding Cayley graph starting at this point, such that these rays are pairwise disjoint. The class corresponding to $[n^3] \times [n^3]$ will then be the boundary of the sum of these rays.

More precisely: Consider the lexicographic ordering $<$ on $[n^3] \times [n^3]$. We inductively define a sequence $(T_{(a,b)})_{a,b \in [n^3]}$ of pairwise disjoint subsets in $\{(x, y) \in (\mathbb{Z}^2)^2 \mid d(x, y) = 1\}$ such that for all $(a, b) \in [n^3] \times [n^3]$

$$\partial_1^{\mathrm{uf}} \chi_{T_{(a,b)}} = (a, b).$$

Then $T := \sum_{a,b \in [n^3]} \chi_{T_{(a,b)}}$ is an element in $C_1^{\mathrm{uf}}(\mathbb{Z}^2; \mathbb{Z})$ and $\partial_1^{\mathrm{uf}} T = \chi_{[n^3] \times [n^3]}$. First, set $T_{(1,1)} = \{((1, r), (1, r + 1)) \mid r \in \mathbb{N}_{>0}\}$. Assume now that for $(a, b) \in [n^3] \times [n^3]$ we already have constructed $T_{(a',b')}$ for all $(a', b') \in ([n^3] \times [n^3])_{<(a,b)}$.

Then we set

$$\begin{aligned} T_{(a,b)} :=& \big\{(a+s, b+s), (a+s+1, b+s+1)) \;\big|\; s \in \{0, \dots, c_{(a,b)} - 1\}\big\} \\ & \cup \big(T_{(1,1)} + (a + c_{(a,b)} - 1, b + c_{(a,b)} - 1)\big) \end{aligned}$$

where

$$c_{(a,b)} := \min\Big\{ s \in \mathbb{N} \;\Big|\; ((k+s, l+s), (k+s, l+s+1)) \notin \bigcup_{\substack{(a',b') \in [n^3] \times [n^3] \\ (a',b') < (a,b)}} T_{(a',b')} \Big\}.$$

So, roughly speaking, the $T_{(a,b)}$ are built by going diagonally up and to the right until one finds a vacant vertical line and then going straight up, Figure 6.5.

By construction, for all $(a, b) \in [n^3] \times [n^3]$

$$c_{(a,b)} \leq c_{(0,b)}.$$

Furthermore, for all $s^3 \in [n^3]$ there are s^2 many points of $[n^3]$ in the square $\{0, 1, \dots, s^3\} \times \{0, 1, \dots, s^3\}$, i.e., we have $|[n^3] \cap \{0, 1, \dots, s^3\}^2| = s^2$. Hence, since the rays are only going up and to the right:

$$c_{(0,s^3)} \leq s^2 < (s+1)^3 - s^3. \tag{$*$}$$

Now the vertical parts of the rays are disjoint by definition and the diagonal parts are disjoint because they are parallel for the same starting value of b and disjoint for different starting values of b by $(*)$. $\square$

Appendix A

Strong Contractions

The core of Ivanov's proof of the absolute mapping theorem [48, Theorem 4.1] is his construction of a norm non-increasing cochain contraction for the cochain complex $B(C_*^{\text{sing}}(X), \mathbb{R})$ for X a simply connected CW-complex. This has been extended by Frigerio and Pagliantini [40] to a relative version for certain CW-pairs. In this section, we will recall the proofs of these results and extend them to coefficients in V', where V can be an arbitrary Banach module. As noted by Bühler [22], it is not necessary to demand X to be countable when an appropriate version of the theorem of Dold and Thom is used, and we will also make use of this observation here.

Lemma A.1 ([48])**.** Let G be an Abelian topological group. For $n \in \mathbb{N}$, the set $S_n(G)$ of singular n-simplices is also an Abelian group. For each $n \in \mathbb{N}$ and $i \in \{0, \dots, n\}$, let $\ell^\infty(\partial^G_{i,n}, \mathbb{R})\colon \ell^\infty(S_{n-1}(G); \mathbb{R}) \longrightarrow \ell^\infty(S_n(G); \mathbb{R})$ be the dual of the i-th face map of $S_n(G)$. Then there is a sequence $(m^n_{\mathbb{R}})_{n\in\mathbb{N}}$ of maps, such that

(i) For each $n \in \mathbb{N}$, the map $m^n_{\mathbb{R}}\colon \ell^\infty(S_n(G), \mathbb{R}) \longrightarrow \mathbb{R}$ is an $S_n(G)$-invariant mean.

(ii) For all $n \in \mathbb{N}_{>0}$ and $i \in \{0, \dots, n\}$

$$\ell^\infty(\partial^G_{i,n}, \mathbb{R})^* m^n_{\mathbb{R}} = m^{n-1}_{\mathbb{R}},$$

where $\ell^\infty(\partial^G_{i,n}, \mathbb{R})^* m^n_{\mathbb{R}}$ denotes the pull-back mean of $m^n_{\mathbb{R}}$ under the face map $\ell^\infty(\partial^G_{i,n}, \mathbb{R})$.

Proof. For each $n \in \mathbb{N}$, the symmetric group Σ_n on n-letters acts on Δ^n by permutation of the coordinates, hence also on $\ell^\infty(S_n(G), \mathbb{R})$. We denote by $M_n \subset \ell^\infty(S_n(G), \mathbb{R})'$ the set of Σ_n-and $S_n(G)$-invariant means. Since $S_n(G)$ is Abelian and thus amenable, there is an $S_n(G)$-invariant mean m on $S_n(G)$, and we can find an $S_n(G)$- and Σ_n-invariant mean by setting

$$m' := \frac{1}{n!} \cdot \sum_{\sigma\in\Sigma_n} \sigma \cdot m.$$

The set M_n is a weak-$*$-closed subset of the unit ball of $\ell^\infty(S_n(G), \mathbb{R})$ and therefore weak-$*$-compact by the Banach-Alaoglu theorem. For any $i \in \{0, \dots, n\}$, the map $\ell^\infty(\partial^G_{i,n}, \mathbb{R})$ induces a weak-$*$-continuous map $f_n\colon M_n \longrightarrow M_{n-1}$. For each $n \in \mathbb{N}_{>0}$, $i, j \in \{0, \dots, n\}$ and $m \in M_n$, we have

$$\ell^\infty(\partial^G_{i,n}, \mathbb{R})^* m = \ell^\infty(\partial^G_{j,n}, \mathbb{R})^* m$$

by the Σ_n-invariance of m, thus f_n does not depend on i and for all $m \in M_n$ and $i \in \{0, \dots, n\}$, we have $\ell^\infty(\partial^G_{i,n}, \mathbb{R})^* m = f_n(m)$. Since the M_n are compact and non-empty, the limit of the directed system $(f_*\colon M_* \longrightarrow M_{*-1})_{*\in\mathbb{N}_{>0}}$ is non-empty. Any point in this limit is a sequence of means which satisfies the desired properties. □

Corollary A.2 ([54])**.** Let G be an Abelian topological group and V a Banach space. Then there is a sequence $(m^n_V)_{n\in\mathbb{N}}$ of maps, such that

(i) For each $n \in \mathbb{N}$, the map $m^n_V\colon \ell^\infty(S_n(G), V') \longrightarrow V'$ is an $S_n(G)$-equivariant mean with coefficients in V', viewed as a trivial $S_n(G)$-module.

(ii) For all $n \in \mathbb{N}_{>0}$ and $i \in \{0, \dots, n\}$

$$\ell^\infty(\partial^G_{i,n}, V')^* m^n_V = m^{n-1}_V.$$

Proof. Let $(m^n_{\mathbb{R}})_{n\in\mathbb{N}}$ be as in Lemma A.1. For all $n \in \mathbb{N}$, set

$$\begin{aligned} m^n_V\colon \ell^\infty(S_n(G), V') &\longrightarrow V' \\ \varphi &\longmapsto \Big(v \longmapsto m^n_{\mathbb{R}}\big(g \longmapsto \varphi(g)(v)\big)\Big). \end{aligned}$$

The first part follows directly from Lemma A.1 (i). For the second part, we have for all $n \in \mathbb{N}$, $i \in \{0, \dots, n\}$ and all $f \in \ell^\infty(S_{n-1}(G), V')$

$$\begin{aligned} \ell^\infty(\partial^G_{i,n}, V')^* m^n_V(f) &= m^n_V(f \circ \partial^G_{i,n}) \\ &= \Big(v \longmapsto m^n_{\mathbb{R}}\big(g \longmapsto f(\partial^G_{i,n}(g))(v)\big)\Big) \\ &= \Big(v \longmapsto m^n_{\mathbb{R}}\big(\ell^\infty(\partial^G_{i,n}, \mathbb{R})\big(g \longmapsto f(g)(v)\big)\big)\Big) \\ &= \Big(v \longmapsto m^{n-1}_{\mathbb{R}}\big(g \longmapsto f(g)(v)\big)\Big) \\ &= m^{n-1}_V(f). \end{aligned}$$

□

Proposition A.3 (Averaging cochains [48, 54])**.** *Let G be an Abelian topological group and let $p\colon X \longrightarrow Y$ be a principal G-bundle. Let V be Banach module. Then there is a cochain map*

$$A^*\colon B(C^{\text{sing}}_*(X), V') \longrightarrow B(C^{\text{sing}}_*(Y), V'),$$

such that $A^* \circ B(C^{\text{sing}}_*(p), V') = \mathrm{id}_{B(C^{\text{sing}}_*(Y),V')}$ and $\|A^*\|_\infty = 1$.

Proof. The group $S_n(G)$ acts on $S_n(X)$ via pointwise multiplication, i.e.,

$$\begin{aligned} S_n(G) \times S_n(X) &\longrightarrow S_n(X) \\ (\tau, \sigma) &\longmapsto \big(t \longmapsto \tau(t) \cdot \sigma(t)\big). \end{aligned}$$

For each $\sigma \in S_n(Y)$, write $F_\sigma = \{\tau \in S_n(X) \mid p(\tau) = \sigma\}$. For any lift $\widetilde{\sigma}$ of σ, we see that F_σ is just the orbit $S_n(G) \cdot \widetilde{\sigma}$ and since the action of $S_n(G)$ on $S_n(X)$ is free, there is an $S_n(G)$-isomorphism

$$\begin{aligned} \rho_{\widetilde{\sigma}} \colon S_n(G) &\longrightarrow F_\sigma \\ \tau &\longmapsto \tau \cdot \widetilde{\sigma}. \end{aligned}$$

Let $(m_V^n)_{n\in\mathbb{N}}$ be as in Corollary A.2. For each $n \in \mathbb{N}$, set $m_\sigma^n = \ell^\infty(\rho_{\widetilde{\sigma}}, V')^* m_V^n$. By the $S_n(G)$-invariance of m_V^n, this does not depend on the choice of the lift $\widetilde{\sigma}$. Now define for all $n \in \mathbb{N}$

$$\begin{aligned} A^n \colon B(C_*^{\mathrm{sing}}(X), V') &\longrightarrow B(C_*^{\mathrm{sing}}(Y), V') \\ f &\longmapsto \big(\sigma \longmapsto m_\sigma^n(f|_{F_\sigma})\big). \end{aligned}$$

Because each m_σ^n is a mean, we see for all $n \in \mathbb{N}$ that $\|A^n\|_\infty \leq 1$ and

$$A^n \circ B(C_n^{\mathrm{sing}}(p), V') = \mathrm{id}_{B(C_n^{\mathrm{sing}}(Y),V')}.$$

Clearly, for any $n \in \mathbb{N}$, any $i \in \{0,\dots,n\}$, any $\sigma \in S_n(Y)$ and any lift $\widetilde{\sigma}$ the following diagram commutes, where the $\partial_{i,n}^?$ are the corresponding i-th face maps:

$$\begin{tikzcd} F_\sigma \arrow[r, "\partial_{i,n}^X|_{F_\sigma}"] & F_{\partial_i^Y \sigma} \\ S_n(G) \arrow[u, "\rho_{\widetilde{\sigma}}"] \arrow[r, "\partial_{i,n}^G"] & S_{n-1}(G). \arrow[u, "\rho_{\partial_{i,n}^X \widetilde{\sigma}}"'] \end{tikzcd}$$

Thus, Property (ii) from Corollary A.2 implies $m_{\partial_{i,n}\sigma}^{n-1} = \ell^\infty(\partial_{i,n}^X|_{F_\sigma}, V')^* m_\sigma^n$. Hence, A^* is a cochain map, since for all $n \in \mathbb{N}_{>0}$, all $i \in \{0,\dots,n-1\}$, all $f \in B(C_{n-1}(X), V')$ and all $\sigma \in S_n(Y)$

$$\begin{aligned} (\delta_{i,n-1}^Y A^{n-1}(f))(\sigma) &= A^{n-1}(f)(\partial_{i,n}^Y \sigma) \\ &= m_{\partial_{i,n}^Y \sigma}^{n-1}\big(f|_{F_{\partial_{i,n}^Y \sigma}}\big) \\ &= m_\sigma^n\Big(\ell^\infty\big(\partial_{i,n}^X|_{F_\sigma}, V'\big)\big(f|_{F_{\partial_{i,n}^Y \sigma}}\big)\Big) \\ &= m_\sigma^n\big((\delta_X^{i,n-1} f)|_{F_\sigma}\big) \\ &= A^n(\delta_X^{i,n-1} f)(\sigma). \end{aligned}$$

□

Theorem A.4 ([48]). *Let X be a simply connected CW-complex. Then there is a sequence*

$$\cdots \xrightarrow{p_n} X_n \xrightarrow{p_{n-1}} \cdots \xrightarrow{p_2} X_2 \xrightarrow{p_1} X_1 := X,$$

such that for all $n \in \mathbb{N}_{>0}$:

(i) For all $i \in \{0,\dots,n\}$, we have $\pi_i(X_n) = 0$.

(ii) For all $i \in \mathbb{N}_{>n}$, the map $\pi_i(p_n) \colon \pi_i(X_{n+1}) \longrightarrow \pi_i(X_n)$ is an isomorphism.

(iii) The map $p_n \colon X_{n+1} \longrightarrow X_n$ is a principal G_n-bundle, where G_n is a topological Abelian group of type $K(\pi_{n+1}(X), n)$.

We call such a sequence a Dold-Thom-Whitehead tower for X.

Sketch of proof. Start with a Whitehead tower (X'_*, p'_*) [46, Example 4.20]. By the theorem of Dold and Thom [32], or, when X is not countable, by its generalisation by McCord [56], we can find for each $n \in \mathbb{N}_{>0}$ a model of $K(\pi_{n+1}(X), n)$ which is a topological Abelian group. By a short argument comparing universal bundles, one can show that one can replace the maps p'_* by principal $K(\pi_{n+1}(X), n)$-bundles, [48, Lemma 2.3.3]. □

We recall the standard cone construction for highly connected spaces:

Remark A.5. For $n \in \mathbb{N}$, let X be an n-connected topological space and $x \in X$. We want to define a family of maps $(L^x_i \colon S_i(X) \longrightarrow S_{i+1}(X))_{i \in \{0,\dots,n\}}$ such that

(i) For all $y \in S_0(X)$, we have $\partial_{0,0} L^x_0(y) = y$ and $\partial_{1,0} L^x_0(y) = x$.

(ii) For all $k \in \{1, \dots, n\}$ and all $i \in \{1, \dots, k\}$, we have

$$\begin{aligned} \partial_{i,k+1} \circ L^x_k &= L^x_{k-1} \circ \partial_{i-1,k} \\ \partial_{0,k+1} \circ L^x_k &= \mathrm{id}_{S_k(X)} . \end{aligned}$$

We define the L^x_k by induction on k. Since X is connected, we can find for each $y \in S_0(X) \cong X$ a path $L^x_0(y)$ from y to x, and L^x_0 satisfies (i). Assume that $L_1, \dots, L_k$ have been defined for $k \in \{0, \dots, n-1\}$. For each $\sigma \in S_k(X)$ consider $\sigma' \colon \partial\Delta^{k+1} \longrightarrow X$, such that for all $i \in \{1, \dots, k+1\}$

$$\begin{aligned} \sigma'|_{\partial_i \Delta^{k+1}} &= L^k_x(\partial_{i-1}\Delta^k) \\ \sigma'|_{\partial_0 \Delta^{k+1}} &= \sigma. \end{aligned}$$

The map σ' is well-defined by property (ii). Since X is k-connected, there is a $k+1$-simplex $L^x_k(\sigma) \in S_{k+1}(X)$, such that $L^x_k(\sigma)|_{\partial\Delta^{n+1}} = \sigma'$. By definition, L^x_k has the desired properties.

We call such a family L^x_* *an n-partial coning contraction for X with respect to x.*

If L^x_* is such a coning contraction, the family

$$(t^x_k \colon C^{\mathrm{sing}}_k(X) \longrightarrow C^{\mathrm{sing}}_{k+1}(X))_{k \in \{0,\dots,n\}},$$

induced by linearly extending L^x_*, together with

$$\begin{aligned} t^x_{-1} \colon \mathbb{R} &\longrightarrow C^{\mathrm{sing}}_0(X) \\ 1 &\longmapsto x \end{aligned}$$

is *an n-partial chain contraction*, i.e., for all $k \in \{0, \dots, n\}$

$$\partial_{k+1} \circ t^x_k + t^x_{k-1} \circ \partial_n = \mathrm{id}_{C_k(X)}$$

and clearly $\|t^x_k\|_\infty \le 1$ for all $k \in \{-1, \dots, n\}$.

The cone construction is not canonical, but sufficiently concrete to prove further compatibility results:

Lemma A.6 ([48, 40]).

(i) Let X be a simply connected CW-complex. Let (X_*, p_*) be a Dold-Thom-Whitehead tower for X. Let $(x_n)_{n\in\mathbb{N}_{>0}}$ be a sequence of points, such that $x_n \in X_n$ and $p_n(x_{n+1}) = x_n$ for all $n \in \mathbb{N}_{>0}$. Then there exists a sequence $(L_*^{x_n})_{n\in\mathbb{N}_{>0}}$, such that $L_*^{x_n}$ is an n-partial coning contraction for X_n with respect to x_n and for all $n \in \mathbb{N}_{>0}$ and $k \in \{0, \ldots, n\}$

$$S_{k+1}(p_n) \circ L_k^{x_{n+1}} = L_k^{x_n} \circ S_k(p_n).$$

We call such a sequence $(L_*^{x_n})_{n\in\mathbb{N}_{>0}}$ *a partial coning for the tower* (X_*, p_*) *with respect to* x_1.

(ii) Let $i\colon W \longrightarrow X$ be a CW-pair. Assume that W and X are simply connected and that i is a weak equivalence. Let (X_*, p_*) be a Dold-Thom-Whitehead tower. Set $W_n := (p_1 \circ \cdots \circ p_n)^{-1}(W)$. Then $(W_*, p_*|_{W_*})$ is a Dold-Thom-Whitehead tower for W. Let $i_n\colon W_n \longrightarrow X_n$ be the canonical inclusion for each $n \in \mathbb{N}_{>0}$. For each $a_1 \in W$, there exists a partial coning $(L_*^{a_n})_{n\in\mathbb{N}_{>0}}$ for (X_*, p_*) and a partial coning $(K_*^{a_n})_{n\in\mathbb{N}_{>0}}$ for $(W_*, p_*|_{W_*})$, such that for all $n \in \mathbb{N}$

$$S_{n+1}(i_{n+1}) \circ K_n^{a_n} = L_n^{a_n} \circ S_n(i_n).$$

Proof. The first part can be found in Ivanov's article [48], the second one in the article of Frigerio and Pagliantini [40]. □

Theorem A.7 ([48, 54]). *Let X be a simply connected CW-complex and V a Banach module. Then for each $x \in X$ there is a strong pointed cochain contraction*

$$(s_x^*\colon B(C_*^{\text{sing}}(X;\mathbb{R}), V') \longrightarrow B(C_{*-1}^{\text{sing}}(X;\mathbb{R}), V'))_{*\in\mathbb{N}}$$

Proof. We follow Ivanov's proof [48], but with coefficients in V'. Let $(L_*^{x_n})_{n\in\mathbb{N}_{>0}}$ be a partial coning for the Dold-Thom-Whitehead tower (X_*, p_*) for $X_1 = X$ with respect to $x_1 = x$ as in Lemma A.6 (i). For each $n \in \mathbb{N}$ let

$$(t_n^k\colon B(C_k^{\text{sing}}(X_n), V') \longrightarrow B(C_{k-1}^{\text{sing}}(X_n), V'))_{k\in\{1,\ldots,n+1\}}$$

be as in Remark A.5. For each $n \in \mathbb{N}_{>0}$, consider

$$A_n^*\colon B(C_*^{\text{sing}}(X_n), V') \longrightarrow B(C_*^{\text{sing}}(X_{n-1}), V')$$

as in Proposition A.3. For each $n \in \mathbb{N}$, set $p_n^* := B(C_*^{\text{sing}}(p_n), V')$ and define

$$(s_x^*\colon B(C_*^{\text{sing}}(X), V') \longrightarrow B(C_{*-1}^{\text{sing}}(X), V'))_{*\in\mathbb{N}}$$

by setting for all $n \in \mathbb{N}_{>0}$

$$s_x^n := A_2^{n-1} \circ \cdots \circ A_{n+1}^{n-1} \circ t_{n+1}^n \circ p_n^n \circ \cdots \circ p_1^n$$

Then $\|s_x^n\|_\infty \leq 1$, since this holds also for all A_*^n, p_*^n and t_n^n. Since for all $n \in \mathbb{N}$

$$\begin{aligned}
&\delta^{n-1} \circ s_x^n + s_x^{n+1} \circ \delta^n \\
&= \delta^{n-1} \circ A_2^{n-1} \circ \cdots \circ A_{n+1}^{n-1} \circ t_{n+1}^n \circ p_n^n \circ \cdots \circ p_1^n \\
&\quad + A_2^n \circ \cdots \circ A_{n+2}^n \circ t_{n+2}^{n+1} \circ p_{n+1}^{n+1} \circ \cdots \circ p_1^{n+1} \circ \delta^n \\
&= A_2^n \circ \cdots \circ A_{n+1}^n \circ \Big(\delta^{n-1} \circ t_{n+1}^n + A_{n+2}^n \circ t_{n+2}^{n+1} \circ p_{n+1}^{n+1} \circ \delta^n\Big) \circ p_n^n \circ \cdots \circ p_1^n \\
&= A_2^n \circ \cdots \circ A_{n+1}^n \circ \Big(\delta^{n-1} \circ t_{n+1}^n + A_{n+2}^n \circ p_{n+1}^n \circ t_{n+1}^{n+1} \circ \delta^n\Big) \circ p_n^n \circ \cdots \circ p_1^n \\
&= A_2^n \circ \cdots \circ A_{n+1}^n \circ \Big(\delta^{n-1} \circ t_{n+1}^n + t_{n+1}^{n+1} \circ \delta^n\Big) \circ p_n^n \circ \cdots \circ p_1^n \\
&= A_2^n \circ \cdots \circ A_{n+1}^n \circ p_n^n \circ \cdots \circ p_1^n \\
&= \mathrm{id}_{B(C_n(X),V')},
\end{aligned}$$

the family s_x^* is a cochain contraction. ☐

Proposition A.8 ([40]). *Let $i\colon A \hookrightarrow X$ be a pair of connected CW-complexes, such that i is π_1-injective and induces an isomorphism between the higher homotopy groups. Let V be a Banach module. Let $\widetilde{i}\colon \widetilde{A} \longrightarrow \widetilde{X}$ be a fixed inclusion map. Then there exists a family of norm non-increasing, pointed cochain contractions*

$$\left(\begin{matrix} \big(s_x^*\colon B(C_*^{\mathrm{sing}}(\widetilde{X}), V') \longrightarrow B(C_{*-1}^{\mathrm{sing}}(\widetilde{X};\mathbb{R}), V')\big)_{*\in\mathbb{N}_{>0}} \\ s_x^0\colon B(C_0^{\mathrm{sing}}(\widetilde{X};\mathbb{R}), V') \longrightarrow V' \end{matrix}\right)_{x\in\widetilde{X}}$$

and a family of norm non-increasing, pointed cochain contractions

$$\left(\begin{matrix} \big(\hat{s}_a^*\colon B(C_*^{\mathrm{sing}}(\widetilde{A};\mathbb{R}), V') \longrightarrow B(C_{*-1}^{\mathrm{sing}}(\widetilde{A};\mathbb{R}), V')\big)_{*\in\mathbb{N}_{>0}} \\ \hat{s}_a^0\colon B(C_0^{\mathrm{sing}}(\widetilde{A};\mathbb{R}), V') \longrightarrow V' \end{matrix}\right)_{a\in\widetilde{A}}$$

that is compatible with the restriction to $\widetilde{A}$, i.e., the following diagram commutes for all $a \in \widetilde{A}$:

$$\begin{tikzcd}
B(C_*^{\mathrm{sing}}(\widetilde{X}), V') \arrow[r, "s^*_{\widetilde{i}(a)}"] \arrow[d, "B(C_*^{\mathrm{sing}}(\widetilde{i}), V')"'] & B(C_{*-1}^{\mathrm{sing}}(\widetilde{X}), V') \arrow[d, "B(C_{*-1}^{\mathrm{sing}}(\widetilde{i}), V')"] \\
B(C_*^{\mathrm{sing}}(\widetilde{A}), V') \arrow[r, "\hat{s}_a^*"] & B(C_{*-1}^{\mathrm{sing}}(\widetilde{A}), V')
\end{tikzcd}$$

Proof. The proof is the same as the one of Theorem A.7, applied to the compatible partial conings from Lemma A.6 (ii). ☐

Remark A.9. The condition on the higher homotopy groups is needed in order to have compatible Dold-Thom-Whitehead towers for the pair (X, A), since under this condition one can use *the same averaging constructions* in the definition of both cochain contractions. If the condition on the higher homotopy groups is not satisfied, one could try to use a functorial version of the Whitehead towers, but then the relation between the averaging maps is completely unclear. If one tries instead to use the preimages $A_n := (p_1 \circ \cdots p_n)^{-1}(A)$ of A inside the

Whitehead tower over X, one has compatible averaging maps, but as noted by Pagliantini [66, Remark 2.29] in general it is impossible to use a relative cone construction with respect to the pair (X_n, A_n), since the homotopy groups of A_n will be in general non-trivial.

Bibliography

[1] C. Anantharaman and J. Renault. Amenable groupoids. In *Groupoids in analysis, geometry, and physics (Boulder, CO, 1999)*, volume 282 of *Contemp. Math.*, pages 35–46. Amer. Math. Soc., Providence, RI, 2001.

[2] C. Anantharaman-Delaroche and J. Renault. *Amenable groupoids*, volume 36 of *Monographies de L'Enseignement Mathématique*. L'Enseignement Mathématique, Geneva, 2000. With a foreword by Georges Skandalis and Appendix B by E. Germain.

[3] M. Aschenbrenner, S. Friedl, and H. Wilton. 3-manifold groups. *ArXiv e-prints:1205.0202*, May 2012.

[4] G. Baumslag, C. Miller, and H. Short. Isoperimetric inequalities and the homology of groups. *Invent. Math.*, 113(3):531–560, 1993.

[5] C. Bavard. Longueur stable des commutateurs. *Enseign. Math. (2)*, 37(1-2):109–150, 1991.

[6] I. Belegradek. Aspherical manifolds with relatively hyperbolic fundamental groups. *Geom. Dedicata*, 129:119–144, 2007.

[7] R. Benedetti and C. Petronio. *Lectures on hyperbolic geometry*. Universitext. Springer-Verlag, Berlin, 1992.

[8] M. Bestvina and K. Fujiwara. Bounded cohomology of subgroups of mapping class groups. *Geom. Topol.*, 6:69–89 (electronic), 2002.

[9] M. Blank and F. Diana. Uniformly finite homology and amenable groups. *ArXiv e-prints:1309.6097*, September 2013. To appear in *Algebraic & Geometric Topology*.

[10] J. Block and S. Weinberger. Aperiodic tilings, positive scalar curvature and amenability of spaces. *J. Amer. Math. Soc.*, 5(4):907–918, 1992.

[11] M. R. Bridson and A. Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, 1999.

[12] J. Brodzki, G. A. Niblo, and N. Wright. Pairings, duality, amenability and bounded cohomology. *J. Eur. Math. Soc. (JEMS)*, 14(5):1513–1518, 2012.

[13] R. Brooks. Some remarks on bounded cohomology. In *Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference (State Univ. New York, Stony Brook, N.Y., 1978)*, volume 97 of *Ann. of Math. Stud.*, pages 53–63. Princeton Univ. Press, Princeton, N.J., 1981.

[14] R. Brooks and C. Series. Bounded cohomology for surface groups. *Topology*, 23(1):29–36, 1984.

[15] K. S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1982.

[16] R. Brown. *Topology and groupoids*. BookSurge, LLC, Charleston, SC, 2006. Third edition of *Elements of modern topology*.

[17] M. Bucher. Simplicial volume of products and fiber bundles. In *Discrete groups and geometric structures*, volume 501 of *Contemp. Math.*, pages 79–86. Amer. Math. Soc., Providence, RI, 2009.

[18] M. Bucher, M. Burger, R. Frigerio, A. Iozzi, C. Pagliantini, and M. B. Pozzetti. Isometric embeddings in bounded cohomology. *ArXiv e-prints:1305.2612*, May 2013.

[19] M. Bucher, M. Burger, and A. Iozzi. A dual interpretation of the Gromov-Thurston proof of Mostow rigidity and volume rigidity for representations of hyperbolic lattices. In *Trends in harmonic analysis*, volume 3 of *Springer INdAM Ser.*, pages 47–76. Springer, Milan, 2013.

[20] M Bucher-Karlsson. The proportionality constant for the simplicial volume of locally symmetric spaces. *Colloq. Math.*, 111(2):183–198, 2008.

[21] M. Bucher-Karlsson. The simplicial volume of closed manifolds covered by $\mathbb{H}^2 \times \mathbb{H}^2$. *J. Topol.*, 1(3):584–602, 2008.

[22] T. Bühler. On the algebraic foundations of bounded cohomology. *Mem. Amer. Math. Soc.*, 214(1006):xxii+97, 2011.

[23] M. Burger and A. Iozzi. Bounded differential forms, generalized Milnor-Wood inequality and an application to deformation rigidity. *Geom. Dedicata*, 125:1–23, 2007.

[24] M. Burger and N. Monod. Bounded cohomology of lattices in higher rank Lie groups. *J. Eur. Math. Soc. (JEMS)*, 1(2):199–235, 1999.

[25] D. Calegari. *scl*, volume 20 of *MSJ Memoirs*. Mathematical Society of Japan, Tokyo, 2009.

[26] T. Ceccherini-Silberstein and M. Coornaert. *Cellular automata and groups*. Springer Verlag, 2010. Springer Monographs in Mathematics.

[27] I. Chatterji, T. Fernós, and A. Iozzi. The median class and superrigidity of actions on cat(0) cube complexes. *ArXiv e-prints:1212.1585*, December 2014.

[28] C. Chou. The exact cardinality of the set of invariant means on a group. *Proc. Amer. Math. Soc.*, 55(1):103–106, 1976.

[29] M. W. Davis, T. Januszkiewicz, and S. Weinberger. Relative hyperbolization and aspherical bordisms: an addendum to "Hyperbolization of polyhedra" [J. Differential Geom. **34** (1991), no. 2, 347–388; MR1131435 (92h:57036)] by Davis and Januszkiewicz. *J. Differential Geom.*, 58(3):535–541, 2001.

[30] F. Diana. *Aspects of uniformly finite homology.* PhD thesis, Universität Regensburg, 2014.

[31] F. Diana and P. W. Nowak. Eilenberg swindles and higher large scale homology of products. *ArXiv e-prints:1409.5219*, September 2014.

[32] A. Dold and R. Thom. Quasifaserungen und unendliche symmetrische Produkte. *Ann. of Math. (2)*, 67:239–281, 1958.

[33] A. Dranishnikov. Macroscopic dimension and essential manifolds. *Tr. Mat. Inst. Steklova*, 273(Sovremennye Problemy Matematiki):41–53, 2011.

[34] A. Dranishnikov. On macroscopic dimension of rationally essential manifolds. *Geom. Topol.*, 15(2):1107–1124, 2011.

[35] A. Dranishnikov. On macroscopic dimension of universal coverings of closed manifolds. *Trans. Moscow Math. Soc.*, pages 229–244, 2013.

[36] T. Dymarz. Bilipschitz equivalence is not equivalent to quasi-isometric equivalence for finitely generated groups. *Duke Math. J.*, 154(3):509–526, 2010.

[37] D. Epstein and K. Fujiwara. The second bounded cohomology of word-hyperbolic groups. *Topology*, 36(6):1275–1289, 1997.

[38] B. Evans and L. Moser. Solvable fundamental groups of compact 3-manifolds. *Trans. Amer. Math. Soc.*, 168:189–210, 1972.

[39] R. Frigerio. (Bounded) continuous cohomology and Gromov's proportionality principle. *Manuscripta Math.*, 134(3-4):435–474, 2011.

[40] R. Frigerio and C. Pagliantini. Relative measure homology and continuous bounded cohomology of topological pairs. *Pacific J. Math.*, 257(1):91–130, 2012.

[41] K. Fujiwara and K. Ohshika. The second bounded cohomology of 3-manifolds. *Publ. Res. Inst. Math. Sci.*, 38(2):347–354, 2002.

[42] S. M. Gersten. Bounded cocycles and combings of groups. *Internat. J. Algebra Comput.*, 2(3):307–326, 1992.

[43] S. M. Gersten. A cohomological characterization of hyperbolic groups. Unpublished, available online via `www.math.utah.edu/~sg/Papers/ch.pdf`, 1996.

[44] S. M. Gersten. Cohomological lower bounds for isoperimetric functions on groups. *Topology*, 37(5):1031–1072, 1998.

[45] M. Gromov. Volume and bounded cohomology. *Inst. Hautes Études Sci. Publ. Math.*, (56):5–99 (1983), 1982.

[46] A. Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[47] P. J. Hilton and U. Stammbach. *A course in homological algebra*, volume 4 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition edition, 1997.

[48] N. V. Ivanov. Foundations of the theory of bounded cohomology. *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)*, 143:69–109, 177–178, 1985. Studies in topology, V.

[49] B. E. Johnson. *Cohomology in Banach algebras.* American Mathematical Society, Providence, R.I., 1972. Memoirs of the American Mathematical Society, No. 127.

[50] J. Kelley and I. Namioka. *Linear topological spaces.* Springer-Verlag, New York-Heidelberg, 1976. Second corrected printing, Graduate Texts in Mathematics, No. 36.

[51] S. Kim and T. Kuessner. Simplicial volume of compact manifolds with amenable boundary. *ArXiv e-prints:1205.1375*, May 2012.

[52] B. Kleiner and B. Leeb. Groups quasi-isometric to symmetric spaces. *Comm. Anal. Geom.*, 9(2):239–260, 2001.

[53] C. Löh. Measure homology and singular homology are isometrically isomorphic. *Math. Z.*, 253(1):197–218, 2006.

[54] C. Löh. *ℓ^1-Homology and Simplicial Volume.* PhD thesis, Universität Münster, 2007. `http://nbn-resolving.de/urn:nbn:de:hbz:6-37549578216`.

[55] J. P. May. *Simplicial objects in algebraic topology.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1992. Reprint of the 1967 original.

[56] M. C. McCord. Classifying spaces and infinite symmetric products. *Trans. Amer. Math. Soc.*, 146:273–298, 1969.

[57] I. Mineyev. Higher dimensional isoperimetric functions in hyperbolic groups. *Math. Z.*, 233(2):327–345, 2000.

[58] I. Mineyev. Straightening and bounded cohomology of hyperbolic groups. *Geom. Funct. Anal.*, 11(4):807–839, 2001.

[59] I. Mineyev. Bounded cohomology characterizes hyperbolic groups. *Q. J. Math.*, 53(1):59–73, 2002.

[60] I. Mineyev, N. Monod, and Y. Shalom. Ideal bicombings for hyperbolic groups and applications. *Topology*, 43(6):1319–1344, 2004.

[61] I. Mineyev and A. Yaman. Relative hyperbolicity and bounded cohomology. Unpublished, available online via `www.math.uiuc.edu/~mineyev/math/art/rel-hyp.pdf`, 2006.

[62] N. Monod. *Continuous bounded cohomology of locally compact groups*, volume 1758 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2001.

[63] N. Monod and Y. Shalom. Cocycle superrigidity and bounded cohomology for negatively curved spaces. *J. Differential Geom.*, 67(3):395–455, 2004.

[64] H. J. Munkholm. Simplices of maximal volume in hyperbolic space, Gromov's norm, and Gromov's proof of Mostow's rigidity theorem (following Thurston). In *Topology Symposium, Siegen 1979 (Proc. Sympos., Univ. Siegen, Siegen, 1979)*, volume 788 of *Lecture Notes in Math.*, pages 109–124. Springer, Berlin, 1980.

[65] G. A. Noskov. Bounded cohomology of discrete groups with coefficients (in russian). *Algebra i Analiz*, 2(5):146–164, 1990.

[66] C. Pagliantini. *Relative (continuous) bounded cohomology and simplicial volume of hyperbolic manifolds with geodesic boundary*. PhD thesis, Universitá di Pisa, 2012. `http://etd.adm.unipi.it/t/etd-07112012-101103/`.

[67] C. D. Papakyriakopoulos. On Dehn's lemma and the asphericity of knots. *Ann. of Math. (2)*, 66:1–26, 1957.

[68] H. Park. Relative bounded cohomology. *Topology Appl.*, 131(3):203–234, 2003.

[69] A. Paterson. *Amenability*, volume 29 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1988.

[70] P. Scott. The geometries of 3-manifolds. *Bull. London Math. Soc.*, 15(5):401–487, 1983.

[71] C. Series. An application of groupoid cohomology. *Pacific J. Math.*, 92(2):415–432, 1981.

[72] T. Soma. The Gromov invariant of links. *Invent. Math.*, 64(3):445–454, 1981.

[73] U. Stammbach. On the weak homological dimension of the group algebra of solvable groups. *J. London Math. Soc. (2)*, 2:567–570, 1970.

[74] W. Thurston. The geometry and topology of 3-manifolds. Lecture notes, Princeton. Available online via `http://library.msri.org/books/gt3m/`, 1978-80.

[75] J. Tu. Groupoid cohomology and extensions. *Trans. Amer. Math. Soc.*, 358(11):4721–4747 (electronic), 2006.

[76] C. A. Weibel. *An introduction to homological algebra*, volume 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[77] J. H. C. Whitehead. On 2-spheres in 3-manifolds. *Bull. Amer. Math. Soc.*, 64:161–166, 1958.

[78] K. Whyte. Amenability, bi-Lipschitz equivalence, and the von Neumann conjecture. *Duke Math. J.*, 99(1):93–112, 1999.